POLAR INTEGRALS - ANSWERS

Do the following by changing to polar coordinates.

1. Find the area of one petal of the rose $r = \cos 2\theta$.

$$\int_{-\pi/4}^{\pi/4}\int_0^{\cos 2\theta} r\,dr\,d\theta = 2\int_0^{\pi/4}\int_0^{\cos 2\theta} r\,dr\,d\theta = 2\int_0^{\pi/4} \left.\frac{r^2}{2}\right|_0^{\cos 2\theta} d\theta = \int_0^{\pi/4} \cos^2 2\theta\,d\theta$$
$$= \int_0^{\pi/4} \frac{1+\cos 4\theta}{2}\,d\theta = \left.\frac{\theta}{2} + \frac{\sin 4\theta}{8}\right|_0^{\pi/4} = \frac{\pi}{8}$$

2. Prove that the area of a circle is πr^2 by evaluating $\iint_R dA$ where R is the disk $x^2 + y^2 \le r^2$.

$$\iint_R dA2 = \int_0^{2\pi}\int_0^{r} r\,dr\,d\theta = \int_0^{2\pi} \left.\frac{r^2}{2}\right|_0^{r} d\theta = \int_0^{2\pi} \frac{r^2}{2}\,d\theta = \left.\frac{\theta r^2}{2}\right|_0^{2\pi} = \pi r^2$$

3. Evaluate $\iint_R \sqrt{x^2 + y^2}\,dA$ where R is the disk $x^2 + y^2 \le 1$.

$$\iint_R \sqrt{x^2 + y^2}\,dA = \int_0^{2\pi}\int_0^{1} \sqrt{r^2}\cdot r\,dr\,d\theta = \int_0^{2\pi}\int_0^{1} r^2\,dr\,d\theta = \int_0^{2\pi} \left.\frac{r^3}{3}\right|_0^{1} d\theta$$
$$= \int_0^{2\pi} \frac{1}{3}\,d\theta = \left.\frac{\theta}{3}\right|_0^{2\pi} = \frac{2\pi}{3}$$

4. Find the volume of the solid bounded above by $z = x^2 + y^2 + 1$ and below by the disk $x^2 + y^2 \le 1$.

$$\iint_R \left(x^2 + y^2 + 1\right)dA = \int_0^{2\pi}\int_0^{1} \left(r^2 + 1\right) r\,dr\,d\theta = \int_0^{2\pi}\int_0^{1} \left(r^3 + r\right)dr\,d\theta$$
$$= \int_0^{2\pi} \left.\left(\frac{r^4}{4} + \frac{r^2}{2}\right)\right|_0^{1} d\theta = \int_0^{2\pi} \frac{3}{4}\,d\theta = \left.\frac{3\theta}{4}\right|_0^{2\pi} = \frac{3\pi}{2}$$

5. Find the surface area of the portion of the paraboloid $z = 4 - x^2 - y^2$ that lies above the *xy*-plane

$$\text{surface area } = \iint_R \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\ dA = \iint_R \sqrt{(-2x)^2 + (-2y)^2 + 1}\ dA$$

$$= \iint_R \sqrt{4x^2 + 4y^2 + 1}\ dA = \iint_R \sqrt{4r^2 + 1}\ r dr d\theta = \int_0^{2\pi}\int_0^2 \left(4r^2 + 1\right)^{1/2} r\ dr d\theta$$

$$= \frac{1}{8}\int_0^{2\pi}\int_1^{17} u^{1/2}\ du d\theta = \int_0^{2\pi} \left.\frac{u^{3/2}}{12}\right|_1^{17} d\theta = \int_0^{2\pi} \frac{17^{3/2} - 1}{12} d\theta = \frac{\pi}{6}\left(17^{3/2} - 1\right) \approx 36.18$$