

LINE INTEGRALS - ANSWERS

1. Evaluate $\int_C xy \, ds$ where C is the unit circle traversed once in the counterclockwise direction.

$$\int_C xy \, ds = \int_a^b x(t)y(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

$$\begin{aligned} x &= \cos t & \frac{dx}{dt} &= -\sin t \\ y &= \sin t & \Rightarrow \frac{dy}{dt} &= \cos t \\ 0 \leq t \leq 2\pi & & \Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} &= \sqrt{\sin^2 t + \cos^2 t} = \sqrt{1} = 1 \end{aligned}$$

$$\int_C xy \, ds = \int_a^b x(t)y(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt = \int_0^{2\pi} \cos t \sin t \, dt = -\int_0^{2\pi} \cos t (-\sin t) \, dt = -\int_1^1 u \, du = 0$$

$$u = \cos t$$

$$du = -\sin t \, dt$$

$$1 \leq u \leq 1$$

2. Evaluate $\int_C xy \, dx$ where C is the unit circle traversed once in the counterclockwise direction.

$$\int_C xy \, dx = \int_a^b x(t)y(t) \frac{dx}{dt} dt$$

$$\begin{aligned} x &= \cos t \\ y &= \sin t & \Rightarrow \frac{dx}{dt} &= -\sin t \\ 0 \leq t \leq 2\pi & & & \end{aligned}$$

$$\int_C xy \, dx = \int_a^b x(t)y(t) \frac{dx}{dt} dt = \int_0^{2\pi} \cos t \sin t (-\sin t) \, dt = \int_0^{2\pi} -\sin^2 t \cos t \, dt = \int_0^0 -u^2 \, du = 0$$

$$u = \sin t$$

$$du = \cos t \, dt$$

$$0 \leq u \leq 0$$

3. Evaluate $\int_C xy \, dy$ where C is the unit circle traversed once in the counterclockwise direction.

$$\int_C xy \, dy = \int_a^b x(t)y(t) \frac{dy}{dt} dt$$

$$\begin{aligned} x &= \cos t \\ y &= \sin t \Rightarrow \frac{dy}{dt} = \cos t \\ 0 &\leq t \leq 2\pi \end{aligned}$$

$$\int_C xy \, dy = \int_a^b x(t)y(t) \frac{dy}{dt} dt = \int_0^{2\pi} \cos t \sin t \cos t dt = \int_0^{2\pi} \cos^2 t \sin t dt = \int_1^{-1} -u^2 du = 0$$

$$\begin{aligned} u &= \cos t \\ du &= -\sin t dt \\ 1 &\leq u \leq 1 \end{aligned}$$

4. Evaluate $\int_C xy \, ds$ where C is the straight line from $(0,0)$ to $(2,4)$.

$$\int_C xy \, ds = \int_a^b x(t)y(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

$$\begin{aligned} x &= 2t \quad \frac{dx}{dt} = 2 \\ y &= 4t \Rightarrow \frac{dy}{dt} = 4 \\ 0 &\leq t \leq 1 \end{aligned} \Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{4+16} = \sqrt{20} = 2\sqrt{5}$$

$$\int_C xy \, ds = \int_a^b x(t)y(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt = \int_0^1 8t^2 \cdot 2\sqrt{5} dt = 16\sqrt{5} \frac{t^3}{3} \Big|_0^1 = \frac{16\sqrt{5}}{3}$$

5. Evaluate $\int_C xy \, dx$ where C is the straight line from $(0,0)$ to $(2,4)$.

$$\int_C xy \, dx = \int_a^b x(t)y(t) \frac{dx}{dt} dt$$

$$\begin{aligned} x &= 2t \\ y &= 4t \Rightarrow \frac{dx}{dt} = 2 \\ 0 &\leq t \leq 1 \end{aligned}$$

$$\int_C xy \, dx = \int_a^b x(t)y(t) \frac{dx}{dt} dt = \int_0^1 8t^2 \cdot 2 dt = \frac{16t^3}{3} \Big|_0^1 = \frac{16}{3}$$

6. Evaluate $\int_C xy \, dy$ where C is the straight line from $(0,0)$ to $(2,4)$.

$$\int_C xy \, dy = \int_a^b x(t)y(t) \frac{dy}{dt} dt$$

$$\begin{aligned} x &= 2t \\ y &= 4t \Rightarrow \frac{dy}{dt} = 4 \\ 0 &\leq t \leq 1 \end{aligned}$$

$$\int_C xy \, dy = \int_a^b x(t)y(t) \frac{dy}{dt} dt = \int_0^1 8t^2 \cdot 4 dt = \frac{32t^3}{3} \Big|_0^1 = \frac{32}{3}$$

7. Evaluate $\int_C x ds$ where C is the curve $y = x^2$ where $0 \leq x \leq 1$.

$$\int_C x ds = \int_a^b x(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

$$\begin{aligned} x &= t & \frac{dx}{dt} &= 1 \\ y &= t^2 \Rightarrow \frac{dy}{dt} = 2t & \Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} &= \sqrt{1 + 4t^2} \\ 0 \leq t &\leq 1 \end{aligned}$$

$$\int_C x ds = \int_a^b x(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt = \int_0^1 t \sqrt{1 + 4t^2} dt = \frac{1}{8} \int_1^5 u^{1/2} du = \frac{1}{12} (5\sqrt{5} - 1)$$

$$\begin{aligned} u &= 1 + 4t^2 \\ du &= 8t dt \\ 1 &\leq u \leq 5 \end{aligned}$$

8. Evaluate $\int_C x dx$ where C is the curve $y = x^2$ where $0 \leq x \leq 1$.

$$\int_C x dx = \int_a^b x(t) \frac{dx}{dt} dt$$

$$\begin{aligned} x &= t \\ y &= t^2 \Rightarrow \frac{dx}{dt} = 1 \\ 0 \leq t &\leq 1 \end{aligned}$$

$$\int_C x dx = \int_a^b x(t) \frac{dx}{dt} dt = \int_0^1 t dt = \frac{t^2}{2} \Big|_0^1 = \frac{1}{2}$$

9. Evaluate $\int_C x dy$ where C is the curve $y = x^2$ where $0 \leq x \leq 1$.

$$\int_C x dy = \int_a^b x(t) \frac{dy}{dt} dt$$

$$x = t$$

$$y = t^2 \Rightarrow \frac{dy}{dt} = 2t$$

$$0 \leq t \leq 1$$

$$\int_C x dy = \int_a^b x(t) \frac{dy}{dt} dt = \int_0^1 2t^2 dt = \left. \frac{2t^3}{3} \right|_0^1 = \frac{2}{3}$$