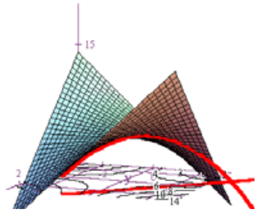
LAGRANGE MULTIPLIERS - ANSWERS

Use the method of Lagrange multipliers to solve the following problems.

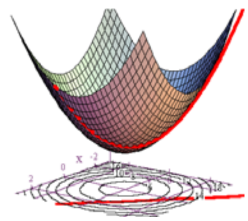


1. Find the coordinates of the maximum point on the graph of $z = xy + 5$ subject to the constraint $x + y = 2$.

$$x + y = 2$$
$$g = x + y$$

$$z_x = y \quad g_x = 1$$
$$z_y = x \quad g_y = 1$$

$$\begin{matrix} z_x = \lambda g_x \\ z_y = \lambda g_y \end{matrix} \Rightarrow \begin{matrix} y = \lambda \\ x = \lambda \end{matrix} \Rightarrow \lambda + \lambda = 2 \Rightarrow \lambda = 1 \Rightarrow \begin{matrix} x = 1 \\ y = 1 \\ z = 6 \end{matrix}$$



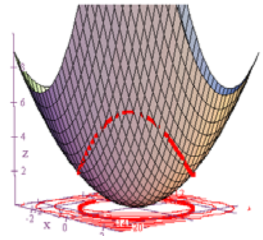
2. Find the coordinates of the minimum point on the graph of $z = x^2 + y^2 + 5$ subject to the constraint $x + y = 2$.

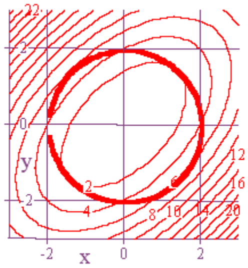
$$x + y = 2$$
$$g = x + y$$

$$z_x = 2x \quad g_x = 1$$
$$z_y = 2y \quad g_y = 1$$

$$\begin{matrix} z_x = \lambda g_x \\ z_y = \lambda g_y \end{matrix} \Rightarrow \begin{matrix} 2x = \lambda \\ 2y = \lambda \end{matrix} \Rightarrow \begin{matrix} x = \lambda/2 \\ y = \lambda/2 \end{matrix} \Rightarrow \frac{\lambda}{2} + \frac{\lambda}{2} = 2 \Rightarrow \lambda = 2 \Rightarrow \begin{matrix} x = 1 \\ y = 1 \\ z = 7 \end{matrix}$$

3. Find the coordinates of the extreme points on the graph of $z = x^2 - xy + y^2$ subject to the constraint $x^2 + y^2 = 4$.





$$x^2 + y^2 = 4$$
$$g = x^2 + y^2$$

$$z_x = 2x - y \quad g_x = 2x$$
$$z_y = 2y - x \quad g_y = 2y$$

$$\begin{matrix} z_x = \lambda g_x \\ z_y = \lambda g_y \end{matrix} \Rightarrow \begin{matrix} 2x - y = 2\lambda x \\ 2y - x = 2\lambda y \end{matrix} \Rightarrow \frac{2x-y}{2y-x} = \frac{2\lambda x}{2\lambda y} \Rightarrow 2xy - x^2 = 2xy - y^2$$
$$\Rightarrow x^2 = y^2 \Rightarrow x^2 + x^2 = 4 \Rightarrow x = \pm\sqrt{2}$$
$$\Rightarrow \left(-\sqrt{2}, -\sqrt{2}\right) or \left(\sqrt{2}, \sqrt{2}\right) or \left(-\sqrt{2}, \sqrt{2}\right) or \left(\sqrt{2}, -\sqrt{2}\right)$$

$\left(-\sqrt{2}, -\sqrt{2}, 2\right)$ is a minimum point

$\left(\sqrt{2}, \sqrt{2}, 2\right)$ is a minimum point

$\left(-\sqrt{2}, \sqrt{2}, 6\right)$ is a maximum point

$\left(\sqrt{2}, -\sqrt{2}, 6\right)$ is a maximum point

4. Let $w = xyz$ for $x \ge 0$, $y \ge 0$, and $z \ge 0$. Find the maximum volume subject to the constraint $x + y + z = 100$.

$$x + y + z = 100$$
$$g = x + y + z$$

$$w_x = yz \quad g_x = 1$$
$$w_y = xz \quad g_y = 1$$
$$w_z = xy \quad g_z = 1$$

$$\begin{matrix} w_x = \lambda g_x \\ w_y = \lambda g_y \\ w_z = \lambda g_z \end{matrix} \Rightarrow \begin{matrix} yz = \lambda \\ xz = \lambda \\ xy = \lambda \end{matrix} \Rightarrow \begin{matrix} \frac{yz}{xz} = \frac{\lambda}{\lambda} = 1 \\ \frac{xz}{xy} = \frac{\lambda}{\lambda} = 1 \end{matrix} \Rightarrow \begin{matrix} y = x \\ z = y \\ x = z \end{matrix} \Rightarrow \begin{matrix} x = \frac{100}{3} \\ y = \frac{100}{3} \\ z = \frac{100}{3} \end{matrix}$$

$$w = \frac{100^3}{3^3} \approx 37{,}037.03704$$

NOTE: We've assumed in our work that $\lambda \ne 0$. Why is this reasonable? What happens to the values of our variables if $\lambda = 0$? What happens if *x*, *y*, or *z* is zero?

5. A manufacturer has an order for 1000 ultra-deluxe time machines with built-in MP3 player. Suppose the units are manufactured in two different locations with *x* representing the number of units produced in one location and *y* the number of units in the other. If the total cost of production is given by $z = C(x, y) = x^2 + 10x + 0.50y^2 + 12y - 10{,}000 \text{ } dollars$, find the values of *x* and *y* that will minimize the costs and find the minimum cost.

$$x + y = 1000 \qquad z_x = 2x + 10 \quad g_x = 1$$
$$g = x + y \qquad z_y = y + 12 \quad g_y = 1$$

$$\begin{matrix} z_x = \lambda g_x \\ z_y = \lambda g_y \end{matrix} \Rightarrow \begin{matrix} 2x + 10 = \lambda \\ y + 12 = \lambda \end{matrix} \Rightarrow \begin{matrix} x = \frac{\lambda - 10}{2} \\ y = \lambda - 12 \end{matrix} \Rightarrow \frac{\lambda - 10}{2} + \lambda - 12 = 1000$$
$$\Rightarrow \lambda - 10 + 2\lambda - 24 = 2000 \Rightarrow 3\lambda = 2034 \Rightarrow \lambda = 678$$

$$x = \frac{678 - 10}{2} = 334$$
$$y = 678 - 12 = 666$$
$$z = 334{,}666$$