

JOINT DENSITY FUNCTIONS - ANSWERS

(1-3) Let $p(x, y) = \begin{cases} \frac{3}{2}x + 3y & \text{if } 0 \leq x \leq 1 \text{ and } 0 \leq y \leq x \\ 0 & \text{elsewhere} \end{cases}$ be a joint density function.

1. Find the probability that $\frac{1}{2} \leq x \leq 1$ and $0 \leq y \leq \frac{1}{2}$.

$$\begin{aligned} \int_{1/2}^1 \int_0^{1/2} p(x, y) dy dx &= \int_{1/2}^1 \int_0^{1/2} \left(\frac{3}{2}x + 3y \right) dy dx = \int_{1/2}^1 \left(\frac{3}{2}xy + \frac{3y^2}{2} \right) \Big|_0^{1/2} dx \\ &= \int_{1/2}^1 \left(\frac{3x}{4} + \frac{3}{8} \right) dx = \left(\frac{3x^2}{8} + \frac{3x}{8} \right) \Big|_{1/2}^1 = \left(\frac{3}{8} + \frac{3}{8} \right) - \left(\frac{3}{32} + \frac{3}{16} \right) \\ &= \frac{15}{32} = 0.46875 \end{aligned}$$

2. Find the probability that $\frac{1}{2} \leq x \leq 1$ and $0 \leq y \leq x$.

$$\begin{aligned} \int_{1/2}^1 \int_0^x p(x, y) dy dx &= \int_{1/2}^1 \int_0^x \left(\frac{3}{2}x + 3y \right) dy dx = \int_{1/2}^1 \left(\frac{3}{2}xy + \frac{3y^2}{2} \right) \Big|_0^x dx \\ &= \int_{1/2}^1 \left(\frac{3x^2}{2} + \frac{3x^2}{2} \right) dx = \int_{1/2}^1 3x^2 dx = x^3 \Big|_{1/2}^1 = 1 - \frac{1}{8} = \frac{7}{8} = 0.875 \end{aligned}$$

3. Find the probability that $0 \leq y \leq \frac{1}{2}$ and $y \leq x \leq \frac{1}{2}$.

$$\begin{aligned} \int_0^{1/2} \int_y^{1/2} p(x, y) dx dy &= \int_0^{1/2} \int_y^{1/2} \left(\frac{3}{2}x + 3y \right) dx dy = \int_0^{1/2} \left(\frac{3x^2}{4} + 3xy \right) \Big|_y^{1/2} dy \\ &= \int_0^{1/2} \left(\frac{3}{16} + \frac{3y}{2} \right) - \left(\frac{3y^2}{4} + 3y^2 \right) dy = \frac{3y}{16} + \frac{3y^2}{4} - \frac{y^3}{4} - y^3 \Big|_0^{1/2} \\ &= \frac{3}{32} + \frac{3}{16} - \frac{1}{32} - \frac{1}{8} = \frac{1}{8} = 0.125 \end{aligned}$$

4. If $p(x)$ is a normal distribution with $\mu=0$ and $\sigma=1$ and if $q(y)$ is another normal distribution with $\mu=0$ and $\sigma=1$, then find the probability that $-1 \leq x \leq 1$ and $-1 \leq y \leq 1$. Set up a double integral and use **fnInt** on your TI-83/84 calculator to approximate numerically rounding to the nearest hundredth.

$$\int_{-1}^1 \int_{-1}^1 \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy dx = \int_{-1}^1 \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \cdot \int_{-1}^1 \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy$$

$$\approx 0.6827 \cdot 0.6827 \approx 0.4661$$

5. If the weights of adult men are normally distributed with a mean of 200 pounds and a standard deviation of 10 pounds, and if IQ is normally distributed with a mean of 100 and a standard deviation of 15 points, then what is the probability that an adult male has a weight between 200 and 210 pounds and an IQ between 100 and 120? Let x equal weight and y equal IQ, set up a double integral, and use **fnInt** on your TI-83/84 calculator to approximate numerically rounding to the nearest hundredth.

$$\int_{200}^{210} \int_{100}^{120} \frac{1}{10\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-200}{10}\right)^2} \cdot \frac{1}{15\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{y-100}{15}\right)^2} dy dx$$

$$= \int_{200}^{210} \frac{1}{10\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-200}{10}\right)^2} dx \cdot \int_{100}^{120} \frac{1}{15\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{y-100}{15}\right)^2} dy \approx 0.3413 \cdot 0.4088 \approx 0.1395$$