

## CYLINDRICAL INTEGRALS - ANSWERS

For each problem below, set up and evaluate a triple integral in cylindrical coordinates.

- Use a triple integral in cylindrical coordinates to find the volume of a cylinder of height  $H$  and radius  $R$ .

$$\begin{aligned} \int_0^{2\pi} \int_0^R \int_0^H r \, dz \, dr \, d\theta &= \int_0^{2\pi} \int_0^R rz \Big|_0^H \, dr \, d\theta = \int_0^{2\pi} \int_0^R rH \, dr \, d\theta = \int_0^{2\pi} \frac{r^2 H}{2} \Big|_0^R \, d\theta \\ &= \int_0^{2\pi} \frac{R^2 H}{2} \, d\theta = \frac{R^2 H}{2} \theta \Big|_0^{2\pi} = \pi R^2 H \end{aligned}$$

- Let  $V$  be a sphere with center at the origin and radius  $= R$ . Find the volume of  $V$ .

$$\begin{aligned} 2 \int_0^{2\pi} \int_0^R \int_0^{\sqrt{R^2-r^2}} r \, dz \, dr \, d\theta &= 2 \int_0^{2\pi} \int_0^R rz \Big|_0^{\sqrt{R^2-r^2}} \, dr \, d\theta = 2 \int_0^{2\pi} \int_0^R r\sqrt{R^2-r^2} \, dr \, d\theta \\ &= -\int_0^{2\pi} \int_{R^2}^0 u^{1/2} \, dud\theta = \int_0^{2\pi} \int_0^{R^2} u^{1/2} \, dud\theta = \int_0^{2\pi} \frac{2u^{3/2}}{3} \Big|_0^{R^2} \, d\theta = \int_0^{2\pi} \frac{2R^3}{3} \, d\theta \\ &= \frac{2R^3}{3} \theta \Big|_0^{2\pi} = \frac{4}{3} \pi R^3 \end{aligned}$$

- Find the volume of the solid bounded by the sphere  $x^2 + y^2 + z^2 = 1$  and the cone  $z = \sqrt{x^2 + y^2}$ .

$$\begin{aligned} \int_0^{2\pi} \int_0^{\frac{1}{\sqrt{2}}} \int_r^{\sqrt{1-r^2}} r \, dz \, dr \, d\theta &= \int_0^{2\pi} \int_0^{\frac{1}{\sqrt{2}}} rz \Big|_r^{\sqrt{1-r^2}} \, dr \, d\theta = \int_0^{2\pi} \int_0^{\frac{1}{\sqrt{2}}} \left( r\sqrt{1-r^2} - r^2 \right) \, dr \, d\theta \\ &= \int_0^{2\pi} \left( \frac{-(1-r^2)^{3/2}}{3} - \frac{r^3}{3} \right) \Big|_0^{\frac{1}{\sqrt{2}}} \, d\theta = \int_0^{2\pi} \left( \frac{-\left(1-\frac{1}{2}\right)^{3/2}}{3} - \frac{\frac{1}{2^{3/2}}}{3} + \frac{1}{3} \right) \, d\theta \\ &= \int_0^{2\pi} \left( -\frac{1}{3\sqrt{2}} + \frac{1}{3} \right) \, d\theta = \left( \frac{1}{3} - \frac{1}{3\sqrt{2}} \right) 2\pi \end{aligned}$$

4. Find the volume of the solid bounded above by  $z = -x^2 - y^2 + 2$  and below by the  $xy$ -plane.

$$\begin{aligned} \int_0^{2\pi} \int_0^{\sqrt{2}} \int_0^{-r^2+2} r \, dz \, dr \, d\theta &= \int_0^{2\pi} \int_0^{\sqrt{2}} r z \Big|_0^{-r^2+2} \, dr \, d\theta = \int_0^{2\pi} \int_0^{\sqrt{2}} (-r^3 + 2r) \, dr \, d\theta \\ &= \int_0^{2\pi} \left( \frac{-r^4}{4} + r^2 \right) \Big|_0^{\sqrt{2}} \, d\theta = \int_0^{2\pi} (-1 + 2) \, d\theta = \int_0^{2\pi} d\theta = 2\pi \end{aligned}$$

5. Find the volume of the solid bounded above by  $z = -x^2 - y^2 + 1$  and below by  $z = x^2 + y^2 - 1$ .

$$\begin{aligned} \int_0^{2\pi} \int_0^1 \int_{r^2-1}^{-r^2+1} r \, dz \, dr \, d\theta &= \int_0^{2\pi} \int_0^1 r z \Big|_{r^2-1}^{-r^2+1} \, dr \, d\theta = \int_0^{2\pi} \int_0^1 (-r^3 + r) - (r^3 - r) \, dr \, d\theta \\ &= \int_0^{2\pi} \int_0^1 (-2r^3 + 2r) \, dr \, d\theta = \int_0^{2\pi} \left( \frac{-2r^4}{4} + r^2 \right) \Big|_0^1 \, d\theta = \int_0^{2\pi} \left( -\frac{1}{2} + 1 \right) \, d\theta \\ &= \int_0^{2\pi} \frac{1}{2} \, d\theta = \frac{\theta}{2} \Big|_0^{2\pi} = \pi \end{aligned}$$