CURVATURE - ANSWERS

For each of the following curves, find the curvature at the indicated value for *t*. Also, state the radius of the best fitting circle at this point $\left(\text{circle radius } = r = \dfrac{1}{\kappa}\right)$. If $\kappa = 0$, then state that $\text{circle radius } = \infty$.

1. $\vec{r}(t) = \cos(t)\hat{i} + \sin(t)\hat{j}$, $0 \le t \le 2\pi$, $t = \dfrac{\pi}{4}$

$$\vec{r}'(t) = -\sin(t)\hat{i} + \cos(t)\hat{j}$$
$$\|\vec{r}'(t)\| = \sqrt{(-\sin t)^2 + (\cos t)^2} = 1$$
$$T(t) = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|} = -\sin(t)\hat{i} + \cos(t)\hat{j}$$
$$T'(t) = -\cos(t)\hat{i} - \sin(t)\hat{j}$$
$$\kappa = \frac{\|T'(t)\|}{\|r'(t)\|} = 1$$
$$\kappa\left(\frac{\pi}{4}\right) = 1$$
$$\text{circle radius } = 1$$

2. $\vec{r}(t) = 2\cos(t)\hat{i} - 2\sin(t)\hat{j}$, $0 \le t \le 2\pi$, $t = \dfrac{5\pi}{4}$

$$\vec{r}'(t) = -2\sin(t)\hat{i} - 2\cos(t)\hat{j}$$
$$\|\vec{r}'(t)\| = \sqrt{(-2\sin t)^2 + (-2\cos t)^2} = 2$$
$$T(t) = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|} = -\sin(t)\hat{i} + \cos(t)\hat{j}$$
$$T'(t) = -\cos(t)\hat{i} - \sin(t)\hat{j}$$
$$\kappa = \frac{\|T'(t)\|}{\|r'(t)\|} = \frac{1}{2}$$
$$\kappa\left(\frac{5\pi}{4}\right) = \frac{1}{2}$$
$$\vec{r}\left(\frac{\pi}{4}\right) = \sqrt{2}\hat{i} - \sqrt{2}\hat{j}$$
$$\text{circle radius } = 2$$

3. $\vec{r}(t) = (2+3t)\hat{i} + (1+4t)\hat{j}$, $0 \le t \le 2$, $t = 1$

$$\vec{r}'(t) = 3\hat{i} + 4\hat{j}$$
$$\|\vec{r}'(t)\| = \sqrt{3^2 + 4^2} = 5$$
$$T(t) = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|} = \frac{3}{5}\hat{i} + \frac{4}{5}\hat{j}$$
$$T'(t) = 0\hat{i} + 0\hat{j} = \vec{0}$$
$$\kappa = \frac{\|T'(t)\|}{\|r'(t)\|} = 0$$
$$\kappa(1) = 0$$

circle radius $= \infty$

4. $\vec{r}(t) = t\,\hat{i} + t^2\,\hat{j}$, $-2 \le t \le 2$, $t = 1$

$$\vec{r}'(t) = \hat{i} + (2t)\hat{j}$$
$$\|\vec{r}'(t)\| = \sqrt{1+4t^2}$$
$$T(t) = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|} = \frac{1}{\sqrt{1+4t^2}}\hat{i} + \frac{2t}{\sqrt{1+4t^2}}\hat{j}$$
$$T'(t) = \frac{-4t}{(1+4t^2)^{3/2}}\hat{i} + \frac{2}{(1+4t^2)^{3/2}}\hat{j}$$
$$T'(1) = \frac{-4}{5^{3/2}}\hat{i} + \frac{2}{5^{3/2}}\hat{j}$$
$$\|T'(1)\| = \sqrt{\frac{16}{5^3} + \frac{4}{5^3}} = \sqrt{\frac{20}{5^3}} = \sqrt{\frac{4}{25}} = \frac{2}{5}$$
$$\vec{r}'(1) = \hat{i} + 2\hat{j}$$
$$\|r'(1)\| = \sqrt{1^2 + 2^2} = \sqrt{5}$$
$$\kappa(1) = \frac{\|T'(1)\|}{\|r'(1)\|} = \frac{2/5}{\sqrt{5}} = \frac{2}{5\sqrt{5}}$$

circle radius $= \dfrac{5\sqrt{5}}{2}$

5. $\vec{r}(t) = \sin t\hat{i} + t\hat{j}$, $0 \le t \le 2\pi$, $t = \pi$

$$\vec{r}'(t) = \cos(t)\hat{i} + \hat{j}$$

$$\|\vec{r}'(t)\| = \sqrt{1+\cos^2 t}$$

$$\|\vec{r}'(\pi)\| = \sqrt{1+\cos^2 \pi} = \sqrt{2}$$

$$T(t) = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|} = \frac{\cos t}{\sqrt{1+\cos^2 t}}\hat{i} + \frac{1}{\sqrt{1+\cos^2 t}}\hat{j}$$

$$T'(t) = \frac{\sqrt{1+\cos^2 t}\,(-\sin t) - \cos t \cdot \frac{1}{2}\left(1+\cos^2 t\right)^{-1/2} \cdot 2\cos t\left(-\sin t\right)}{1+\cos^2 t}\hat{i}$$

$$-\frac{1}{2}\left(1+\cos^2 t\right)^{-3/2} \cdot 2\cos t\left(-\sin t\right)\hat{j}$$

$$T'(\pi) = 0\hat{i} + 0\hat{j} = \vec{0}$$

$$\|T'(\pi)\| = 0$$

$$\kappa(\pi) = \frac{\|T'(\pi)\|}{\|r'(\pi)\|} = \frac{0}{\sqrt{2}} = 0$$

circle radius $= \infty$