CONSTRUCTIONS - ANSWERS

1. Let $z = f(x, y) = x^2 + y^2$. Find parametric equations for the cross-section of this surface with the plane $x = 1$.

$x = 1$
$y = t$
$z = 1 + t^2$
$-\infty < t < \infty$

2. Let $z = f(x, y) = x^2 + y^2$. Find parametric equations for the tangent line at the point $P = (1, 2, 5)$ that lies in the plane $x = 1$.

$z_y = 2y$
$z_y(2) = 4$
$x = 1$
$y = 2 + t$
$z = 5 + 4t$
$-\infty < t < \infty$

3. Let $z = f(x, y) = x^2 + y^2$. Find parametric equations for the cross-section of this surface with the plane $y = 2$.

$x = t$
$y = 2$
$z = t^2 + 4$
$-\infty < t < \infty$

4. Let $z = f(x, y) = x^2 + y^2$. Find parametric equations for the tangent line at the point $P = (1, 2, 5)$ that lies in the plane $y = 2$.

$z_x = 2x$
$z_x(1) = 2$
$x = 1 + t$
$y = 2$
$z = 5 + 2t$
$-\infty < t < \infty$

5. Let $z = f(x, y) = x^2 + y^2$. Find parametric equations for the tangent plane to this surface at the point $P = (1, 2, 5)$.

The parametric equations found in the previous problems for the tangent lines can be used to help us find two non-parallel vectors that lie in the tangent plane. In particular, let,

$$\vec{r} = \hat{i} + 2\hat{j} + 5\hat{k}$$
$$s \cdot \vec{u} = s\hat{j} + 4s\hat{k}$$
$$t \cdot \vec{v} = t\hat{i} + 2t\hat{k}$$

Then,

$$\text{Tangent Plane } = \vec{r} + s\vec{u} + t\vec{v}$$

And this implies,

$$x = 1 + t$$
$$y = 2 + s$$
$$z = 5 + 4s + 2t$$
$$-\infty < s < \infty$$
$$-\infty < t < \infty$$

6. Let $z = f(x, y) = x^2 + y^2$. Find an equation for the tangent plane to this surface at the point $P = (1, 2, 5)$. Write your answer in the form $z = Ax + By + C$.

If we rewrite our function as $0 = x^2 + y^2 - z$, then we can think of this as a level surface for the function $w = x^2 + y^2 - z$ If we evaluate the gradient of w at $P = (1, 2, 5)$, we get $\nabla w = 2x\hat{i} + 2y\hat{j} - \hat{k} \Rightarrow \nabla w(1, 2, 5) = 2\hat{i} + 4\hat{j} - \hat{k}$. This vector is perpendicular to the tangent plane of $z = f(x, y) = x^2 + y^2$ at the point $P = (1, 2, 5)$. If $Q = (x, y, z)$ is another point in the tangent plane, then the displacement vector from P to Q is $\overrightarrow{PQ} = (x-1)\hat{i} + (y-2)\hat{j} + (z-5)\hat{k}$. Hence, $\nabla w(1, 2, 5) \bullet \overrightarrow{PQ} = 0$ implies that $2 \cdot (x-1) + 4 \cdot (y-2) + (-1) \cdot (z-5) = 0 \Rightarrow 2x + 4y - z - 5 = 0 \Rightarrow z = 2x + 4y - 5$.