CHAPTER 9

VECTOR FIELDS

I'll always remember quite vividly how confused I was when I heard that vectors had magnitude and direction but didn't exist at any particular location in space. This was confusing to me because the common interpretation of a vector was that of a force, and forces always acted at particular locations. For example, a river might be raging at one point and the water might be calm at another, and there is no way that the force at the former point can be arbitrarily moved to a different location. It was only much later that I understood that mathematicians didn't force (no pun intended!) vectors to have a location in space because they wanted to be able to move them around like legos or tinker toys so that they could easily be added together by placing the initial point of one vector on top of the terminal point of another. Furthermore, if vectors possessed location, then something like $\vec{v} = 2\hat{i} + 3\hat{j}$ would have to represent a vector that started at one and only one point, and it would, consequently, be very difficult to define vector arithmetic. Nevertheless, the concept of vectors that are tied to points does result in an important model of the real world, and such a model is what we are going to call a <u>vector field</u>.

Technically, we'll say that a vector field is a function that assigns a vector to a point

in *n*-dimensional space. For example, if to each point (x, y, z) we assign the vector

$\vec{F}(x, y, z) = x\hat{i} - y\hat{j} - z\hat{k}$, then a graph of the resulting vector field looks like this.
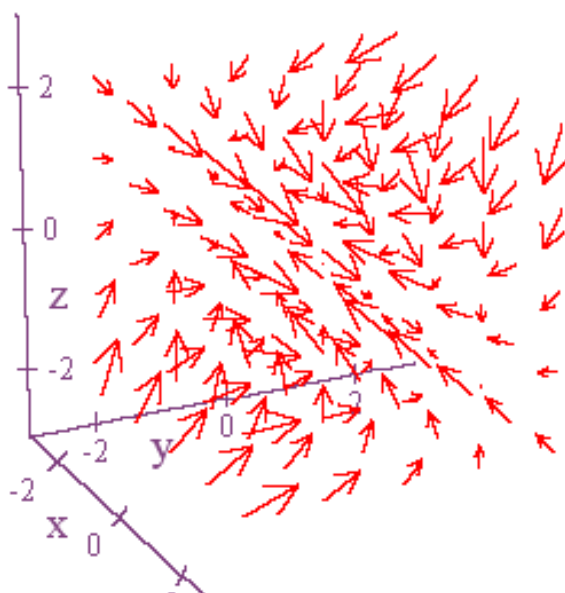

As you can see, however, even using software to draw this thing, it's still a bit of a

mess. Thus, we're going to spend most of our time looking at vectors fields in just

two dimensions. These will be far easier to analyze and understand, and as a result,

they will do a better job of illustrating the concepts we are interested in. Below is a

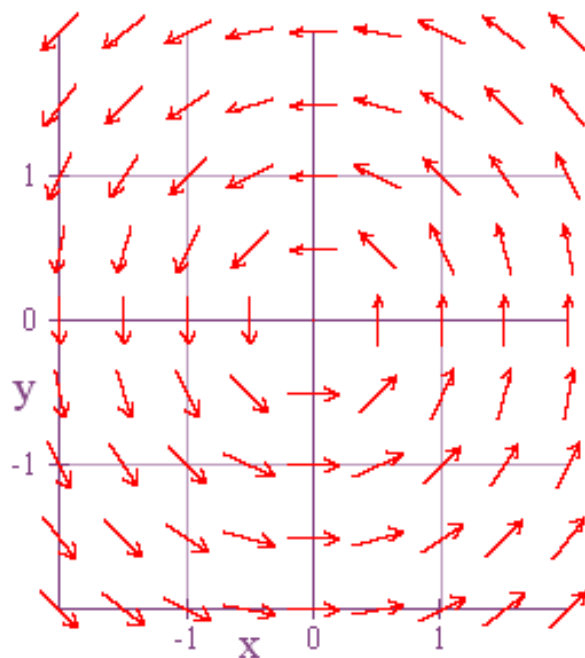graph of the vector field $\vec{F}(x, y) = x\hat{i} + y\hat{j}$.
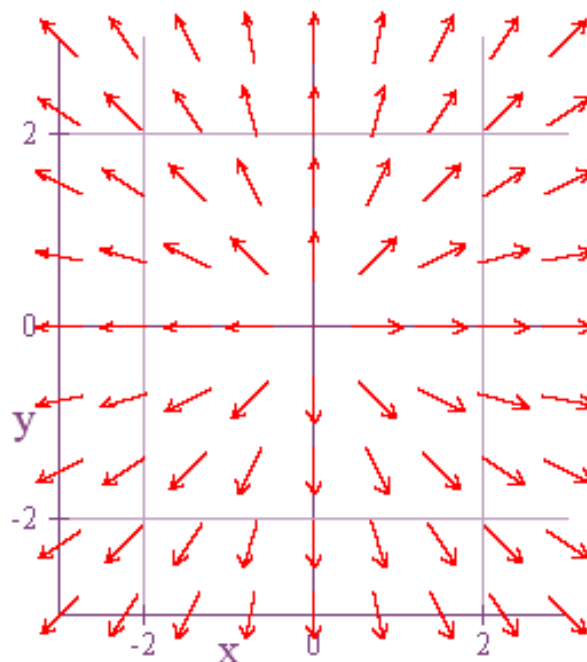
When drawing a vector field, there are a few ways to do it, and it's always a compromise between accuracy and clarity. Technically speaking, the most accurate way to do it would to be to find the vector associated with a point and then draw in that vector by placing the initial point of the vector at our designated point (x, y). In practice, though, the resulting graph is sometimes easier to understand if we draw it so that the midpoint of the vector is at (x, y). Additionally, if we draw each vector with the correct magnitude shown, then things can get messy very quickly. For instance, here is the vector field we just looked at with the true size of each vector shown. As you can see, the result is much harder to understand.
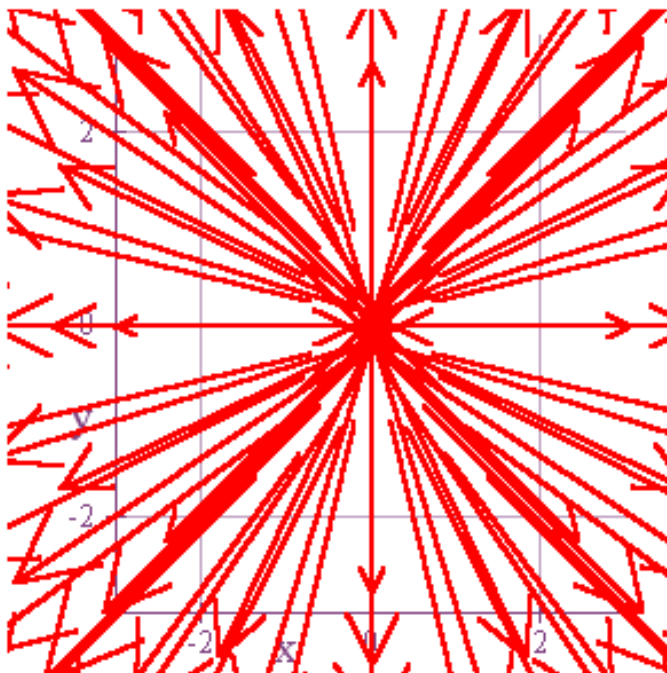
Thus, for the sake of clarity, it's sometimes best to make each vector in your drawing the same length. In practice, this is generally not overly restrictive because we are often interested, primarily, in what we might call the flow pattern of the vectors. In other words, if the vectors represent something like the velocity of a river at each point, then we may simply want to know what would happen if something like a ping pong ball were dropped into the river. Where would the river take it? In the graph of $\vec{F}(x,y) = -y\hat{i} + x\hat{j}$ below, it's easy to understand that the flow of the river would take the ball around in circles.
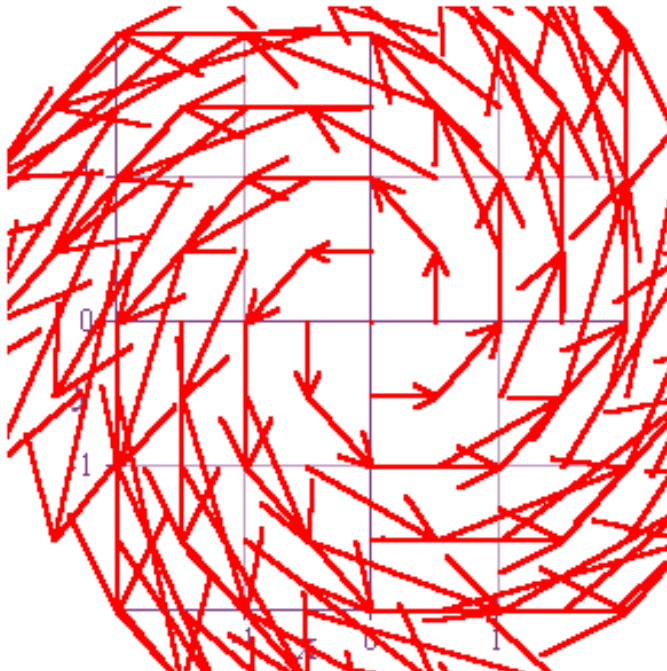
Of course, if we are using software to graph our vector field, then we are always at the mercy of the options available in our program. In the graph above, the point $(1,0)$ gets paired with the vector $\vec{F}(1,0) = -0\hat{i} + 1\hat{j} = \hat{j}$. You can see from the diagram that the vectors are not drawn according to their actual length, and the vector for $(1,0)$ is drawn so that it's midpoint is at the point $(1,0)$. If we adjust things so that the initial point is at $(1,0)$, then our picture looks like this.
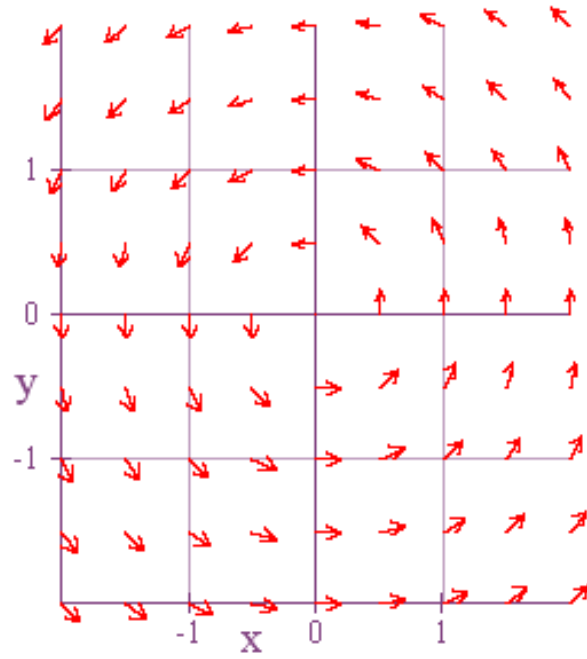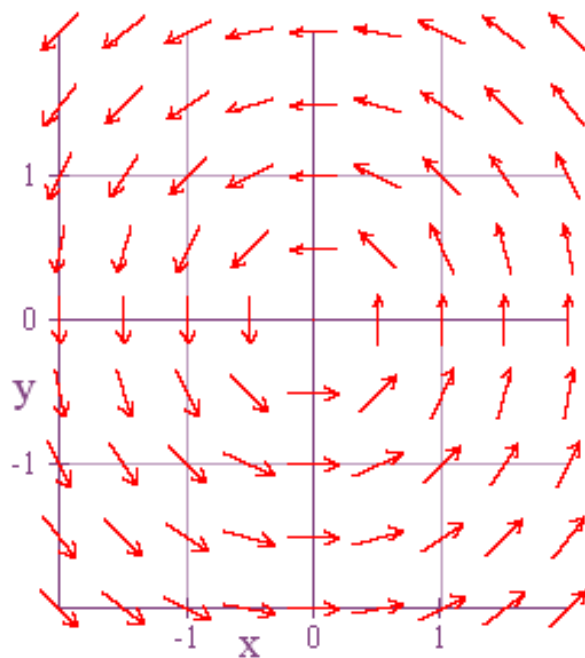
And if we also give each vector its true length, then we get this.

A bit of mess, isn't it! The bottom line is that you want to use the graphing options that give you the information you most want, and for me, our original graph is what is easiest to look at in terms of simply understanding the flow created by the vector field.

 A good thing to do at least once in your life is to take the definition for a vector field and actually create a table of vectors that you might plot. For example, if our vector field is $\vec{F}(x,y) = -y\hat{i} + x\hat{j}$, then a table of values might look like the following.

x	y	$\vec{F}(x,y) = -y\hat{i} + x\hat{j}$
0	0	$\vec{0}$
1	0	$\hat{j}$
0	1	$-\hat{i}$
-1	0	$-\hat{j}$
0	-1	$\hat{i}$
2	3	$-3\hat{i} + 2\hat{j}$
-4	1	$-\hat{i} - 4\hat{j}$

Try plotting these vectors by hand, and you'll understand how the diagrams above are created.

If we think of our vector field again as being a flowing river and if we drop a ping pong ball in the river, then the path that it takes is what we'll call a <u>flow line</u>. For the vector field $\vec{F}(x,y) = -y\hat{i} + x\hat{j}$, it's easy to see that all the flow lines are circular.
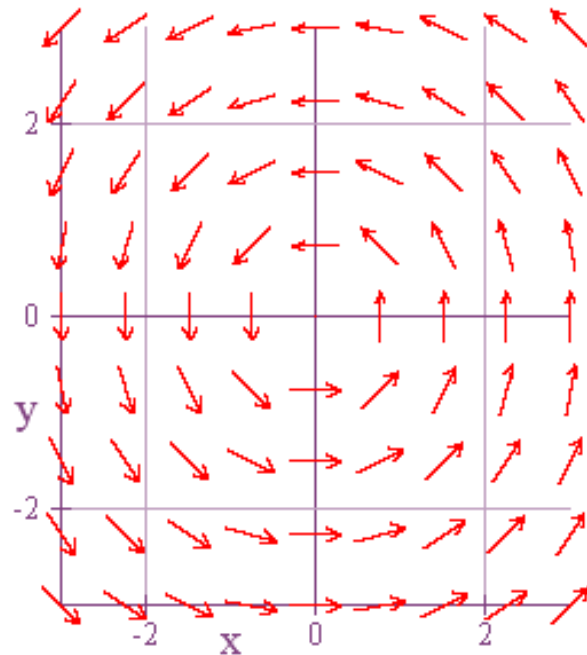
However, we can try and plot a particular flow line by using the following

approximation method due to Euler. First, pick a particular point (a,b). This is

where we'll drop the ping pong ball. Next, find the vector at that point and multiply

it by an increment Δt. In the example I'll show below, I'll set $\Delta t = 0.1$. This means

that I'll multiply the vector at (a,b) by a tenth, and then I'll add this vector to my

point to get my next point, (a_1,b_1). And then I repeat this procedure several times in

order to get several points to plot. To go back in the opposite direction, I just change

the sign of my increment, $-\Delta t = -0.1$. As long as your increment is not very large,

you'll get a fairly decent plot of points along your flow line. However, as Δt

increases in size, your accuracy diminishes. Below is a plot to delineate the flow line

when my starting point is $(2,1)$. The starting point is plotted as a big red spot, and the
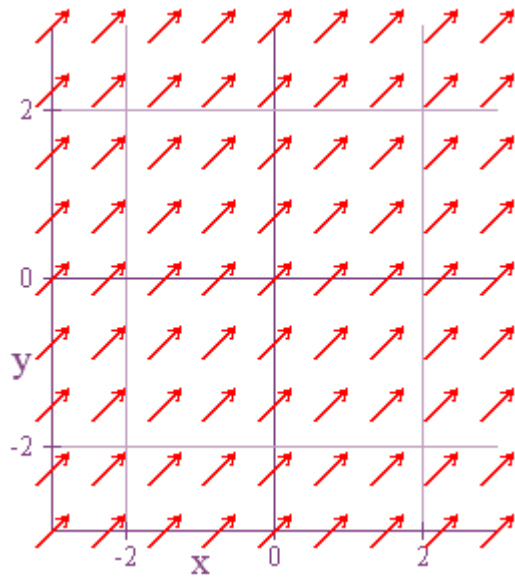
points corresponding to positive increments are plotted in blue while the points

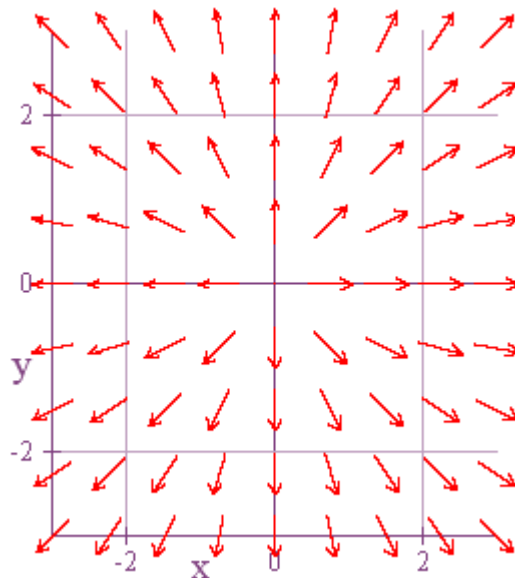resulting from a negative increment are plotted in green. Here's the result.
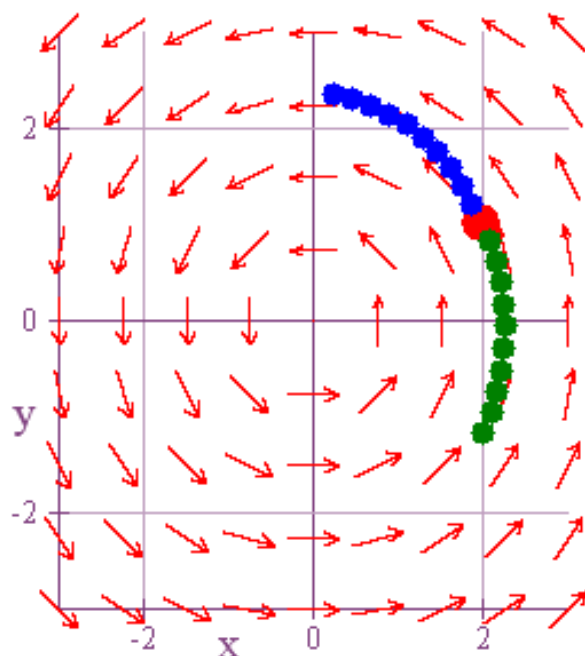
Not bad! Now let's just look at the graphs of a variety of vector fields. As you look

at each one, pick a point and try to visualize how the corresponding vector would be

drawn at that point. That will help you understand the diagrams with greater ease.
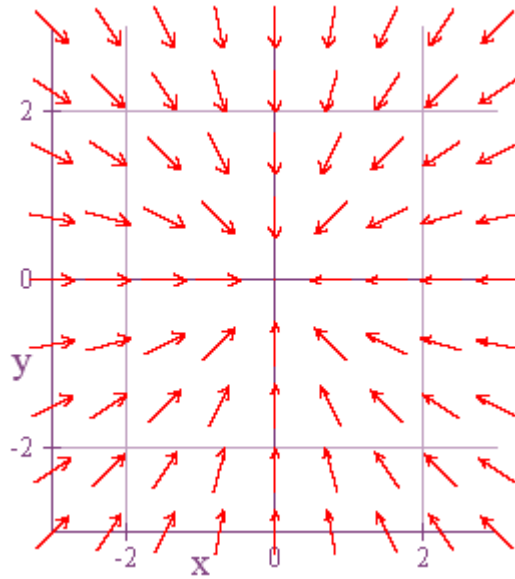
Example 1: $\vec{F}(x, y) = \hat{i} + \hat{j}$

Example 2: $\vec{F}(x, y) = x\hat{i} + y\hat{j}$

Vector Fields
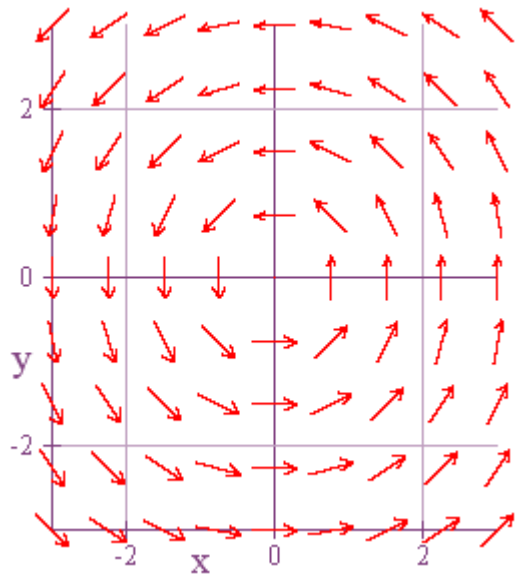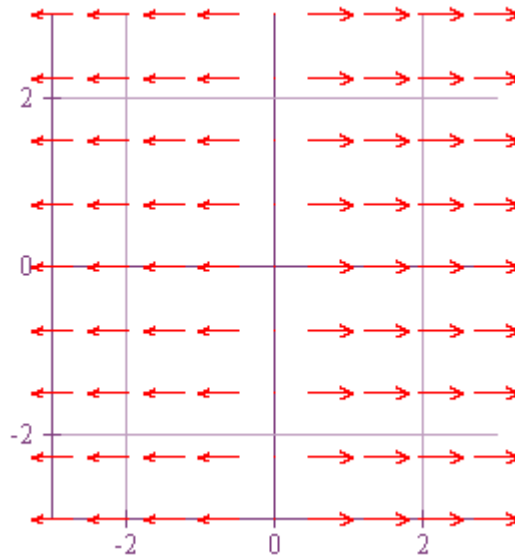
Example 3: $\vec{F}(x,y) = -x\hat{i} - y\hat{j}$

Example 4: $\vec{F}(x,y) = -y\hat{i} + x\hat{j}$

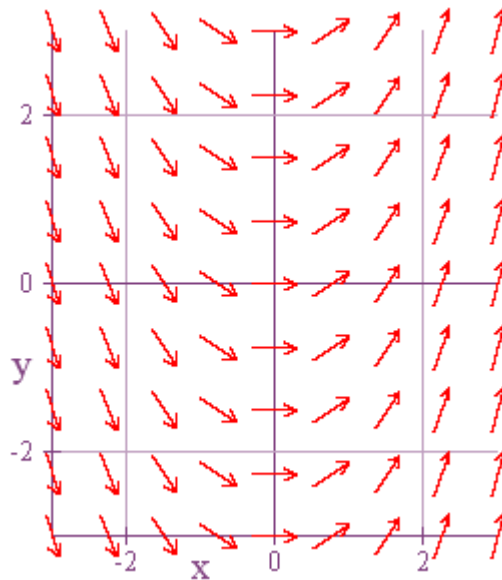Example 5: $\vec{F}(x, y) = x\hat{i} + 0\hat{j} = x\hat{i}$

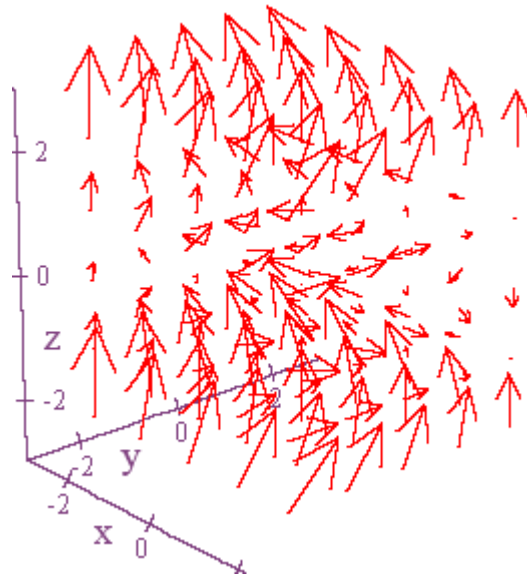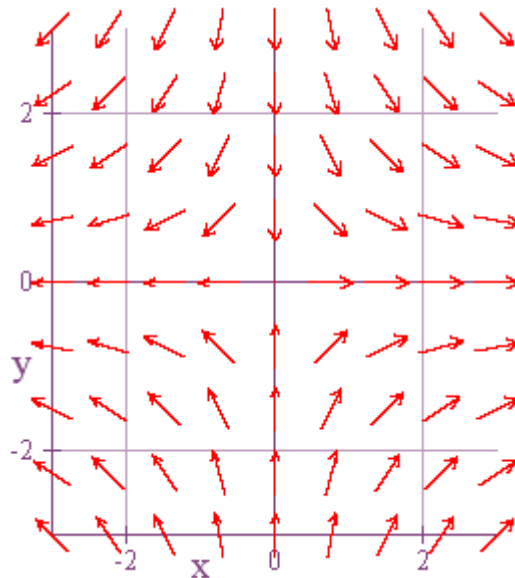Example 6: $\vec{F}(x, y) = \hat{i} + x\hat{j}$

Example 7: $\vec{F}(x,y,z) = x\hat{i} - y\hat{j} + z^2\hat{k}$

Example 8: $z = f(x,y) = x^2 - y^2$, $\nabla f(x,y) = \vec{F}(x,y) = 2x\hat{i} - 2y\hat{j}$

The last example above illustrates something very important. An easy way to get a vector field from a function of several variables is by finding its gradient. The resulting vector field is called a <u>gradient field</u>, and the multivariable function that gives rise to it is called a <u>potential</u> or <u>potential function</u>.

One thing we might notice if we study the vector fields above is that some of them will tend to cause circulation while others will tend to result in flux across a boundary. For example, the vector field below will tend to cause things to circulate counterclockwise around the circle below, but won't tend to move things across the boundary of the circle.
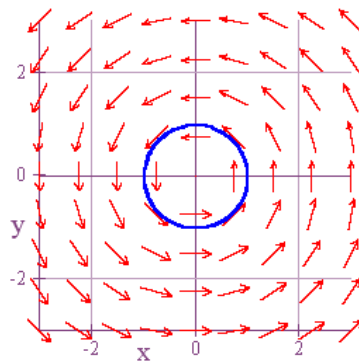
On the other hand, this next vector field will cause a definite outward flux across the

circle's boundary, but it won't create any circulation around the circle.
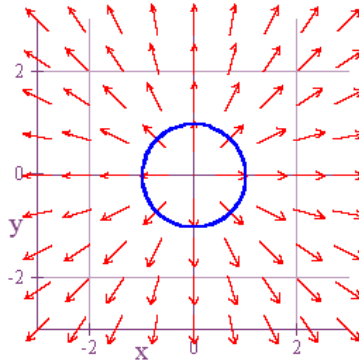
In calculus, there are computations to be done that can measure the tendency of

vector field to create either circulation or flux at a particular point. These calculations

are called, respectively, the <u>curl</u> and the <u>divergence</u> of the vector field. It won't be

apparent until the final chapter of this book why these definitions work, but for now

just accept that they do and learn how to do the computations. The definition of the

curl of a vector field is as follows.

If $\vec{F} = P(x,y,z)\hat{i} + Q(x,y,z)\hat{j} + R(x,y,z)\hat{k}$, then

$$curl\ of\ \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \begin{vmatrix} \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ Q & R \end{vmatrix}\hat{i} - \begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial z} \\ P & R \end{vmatrix}\hat{j} + \begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} \\ P & Q \end{vmatrix}\hat{k}$$

$$= \left(\dfrac{\partial R}{\partial y} - \dfrac{\partial Q}{\partial z}\right)\hat{i} - \left(\dfrac{\partial R}{\partial x} - \dfrac{\partial P}{\partial z}\right)\hat{j} + \left(\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y}\right)\hat{k}$$

If $\vec{F}$ is a 2-dimensional vector field, $\vec{F} = P\hat{i} + Q\hat{j}$, then

$$curl\ of\ \vec{F} = \nabla \times \vec{F} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \hat{k}$$

In the definitions above, the cross product notation provides an easy way to remember how to find the curl of a vector field. The symbol ∇ refers to the gradient, but we don't have a function indicated that we should take the gradient of. In this case, the functions come from the components of $\vec{F}$. Thus, if $\vec{F}(x, y, z) = -y^2\,\hat{i} + 2z\,\hat{j} + x^3\,\hat{k}$, then the curl of $\vec{F}$ is,

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^2 & 2z & x^3 \end{vmatrix} = \left(\frac{\partial(x^3)}{\partial y} - \frac{\partial(2z)}{\partial z} \right)\hat{i} - \left(\frac{\partial(x^3)}{\partial x} - \frac{\partial(-y^2)}{\partial z} \right)\hat{j} + \left(\frac{\partial(2z)}{\partial x} - \frac{\partial(-y^2)}{\partial y} \right)\hat{k}$$

$$= -2\hat{i} - 3x^2\,\hat{j} + 2y\hat{k}$$

Notice a couple of things here. First, the curl of a vector field is another vector field since it is defined by a cross product. And second, since we have variables left in our answer, the curl will generally vary from point to point. In any case, however, this particular vector field will tend to produce circulation. Below is graph of

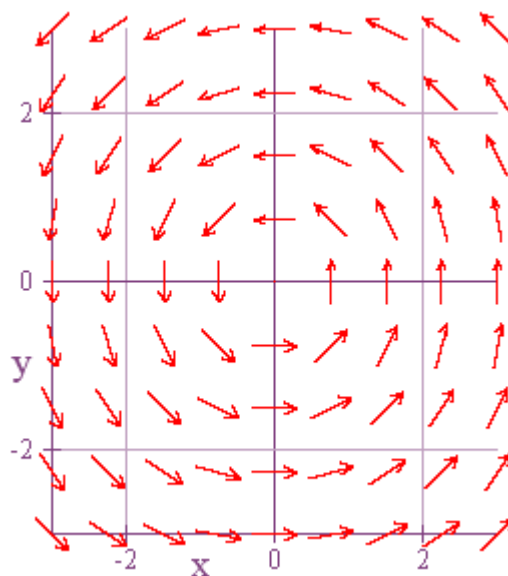$$\vec{F}(x, y, z) = -y^2\,\hat{i} + 2z\,\hat{j} + x^3\,\hat{k}.$$
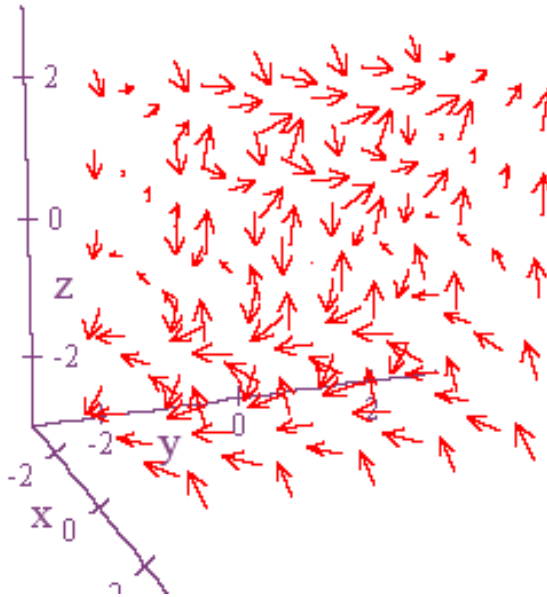
If we look at the two dimensional vector field $\vec{F}(x, y) = -y\hat{i} + x\hat{j}$, then the curl is,

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -y & x & 0 \end{vmatrix} = \left(\dfrac{\partial x}{\partial x} - \dfrac{\partial(-y)}{\partial y} \right) \hat{k} = 2\hat{k}.$$

In this case, we see that the circulation is the same at every point in the field.

The scalar component of the curl of a two dimensional vector field is sometimes referred to as the scalar curl. Also, compare the result above with what happens with the curl of the vector field $\vec{F} = x\hat{i} + y\hat{j}$.

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x & y & 0 \end{vmatrix} = \left(\dfrac{\partial y}{\partial x} - \dfrac{\partial(x)}{\partial y} \right)\hat{k} = 0\hat{k} = \vec{0}$$

The result here is that the scalar curl is zero, i.e. $\nabla \times \vec{F}$ is the zero vector. This corresponds to the fact that the vector field, as plotted below, has no tendency to produce circulation.
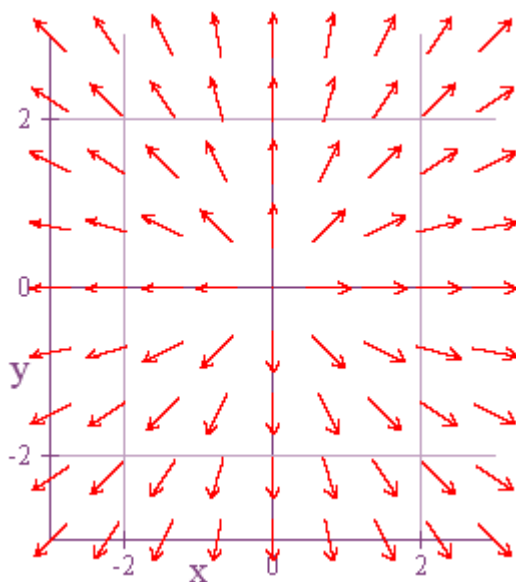
The other computation we want to learn now is the divergence of a vector field. This computation will tell us something about the tendency of a vector field to produce flux across a boundary. We define this quantity as follows.

If $\vec{F} = P(x,y,z)\hat{i} + Q(x,y,z)\hat{j} + R(x,y,z)\hat{k}$, then $divergence\, of\, \vec{F} = \nabla \cdot \vec{F}$

$$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right) \cdot \left(P\hat{i} + Q\hat{j} + R\hat{k} \right) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$

If $\vec{F}$ is a 2-dimensional vector field, $\vec{F} = P\hat{i} + Q\hat{j}$, then $divergence\, of\, \vec{F} = \nabla \cdot \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}$

Notice both that the divergence is easily defined in terms of a dot product and that the final result is a scalar or scalar function, not a vector. Let's use $\vec{F} = x\hat{i} + y\hat{j}$ as an example. In this case, the divergence is,

$$\nabla \bullet \vec{F} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} = 1 + 1 = 2$$
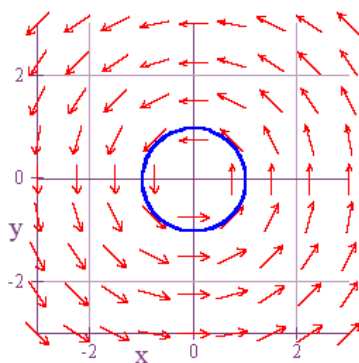
We can see that the result is a nonzero number, and that, in this particular example, the divergence is the same at all points in our plane. Let's now look at a graph of our vector field with a circle drawn about the origin.
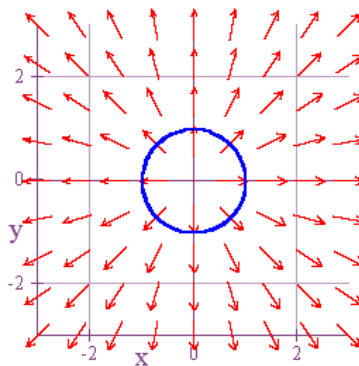
It's very easy to see in this case that the vector field will tend to cause a flux across the boundary of this circle.

Now look at a similar plot for $\vec{F}(x, y) = -y\hat{i} + x\hat{j}$.

Does this vector field appear to be one that would cause flux across this boundary?

Probably not, and so it should be no surprise that the divergence comes out equal to

zero.

$$\nabla \bullet \vec{F} = \frac{\partial(-y)}{\partial x} + \frac{\partial x}{\partial y} = 0 + 0 = 0$$

A final word of caution, however. Even well drawn pictures can sometimes be a little

misleading. Thus, the bottom line is always what the formulas tell us, not what you

think the picture shows. A picture can be correct, but we can still misinterpret it.

Let's also look at two final examples of vector fields in two dimensions that are very

instructive. We'll begin by computing the curl of the vector field $\vec{F} = y\hat{i} - x\hat{j}$.

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -x & 0 \end{vmatrix} = \left(\frac{\partial(-x)}{\partial x} - \frac{\partial y}{\partial y} \right)\hat{k} = -2\hat{k}$$

The scalar curl of this vector field is negative, and this corresponds to the fact that this field will tend to produce circulation in the clockwise rather than the counterclockwise direction. Because of this type of result, we always think of counterclockwise rotation as positive and clockwise as negative. Below is the graph of this vector field along with a circular curve with center at the origin. The vector field will circulate things in the clockwise direction around this curve.
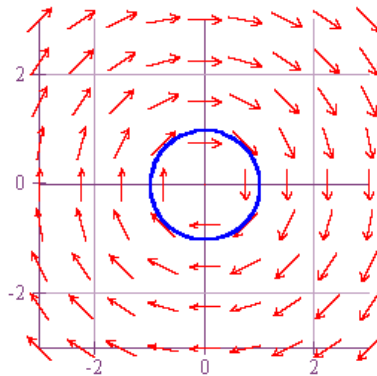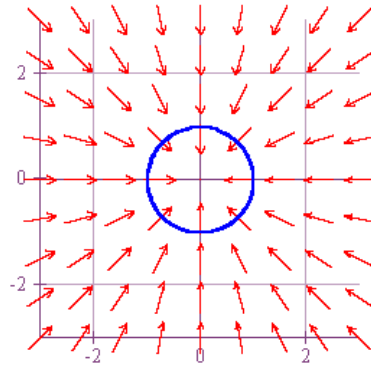

And lastly, let's look at the divergence of the vector field $\vec{F} = -x\hat{i} - y\hat{j}$. The divergence is,

$$\nabla \bullet \vec{F} = \frac{\partial(-x)}{\partial x} + \frac{\partial(-y)}{\partial y} = -1 - 1 = -2$$

In this case, the divergence is negative, and we can see from the graph below how this result corresponds to a flux across our boundary that is towards the center of our circle rather than away from it.

Vector Fields

Next stop, line integrals!