CHAPTER 6

VECTOR CALCULUS

We've spent a lot of time so far just looking at all the different ways you can graph

things and describe things in three dimensions, and it certainly seems like there is a

lot more variety than what we encounter in just two dimensions. So much so that, in

case you haven't noticed, we haven't done a single integral or derivative yet! Well,

maybe we can remedy that in this chapter. Maybe it's just about time to see what we

can do with derivatives and integrals when we apply them to vector-valued functions.

In fact, for starters, let's take a look at the standard parametrization for the unit circle,

$\vec{r}(t) = \cos(t)\hat{i} + \sin(t)\hat{j}$, where $0 \le t \le 2\pi$. If we want to find the derivative of this

function, we have to go back to the basics, the way in which a derivative was defined

for you in your first calculus course. Thus, recall that one of the definitions you may

have been given was $\dfrac{df}{dx} = \lim_{\Delta x \to 0} \dfrac{f(x + \Delta x) - f(x)}{\Delta x}$. If we try something similar with the

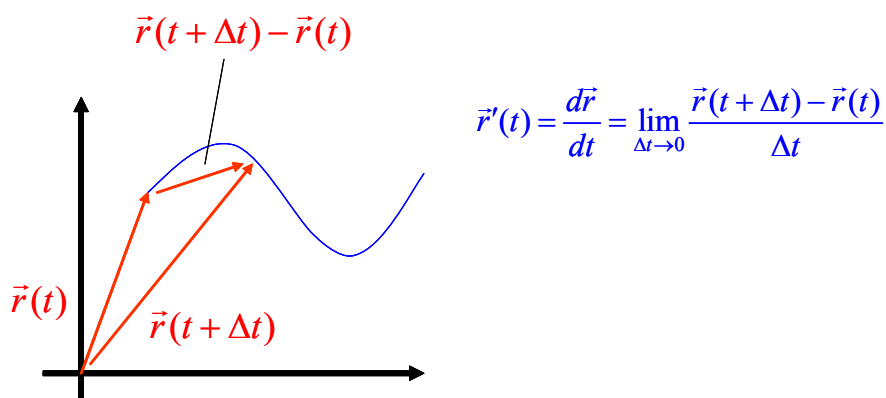function $\vec{r}(t) = \cos(t)\hat{i} + \sin(t)\hat{j}$, then we get

$$\frac{d\vec{r}}{dt} = \lim_{\Delta t \to 0} \frac{\vec{r}(t + \Delta t) - \vec{r}(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\left[\cos(t + \Delta t)\hat{i} + \sin(t + \Delta t)\hat{j}\right] - \left[\cos(t)\hat{i} + \sin(t)\hat{j}\right]}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{\left[\cos(t + \Delta t) - \cos(t)\right]\hat{i} + \left[\sin(t + \Delta t) - \sin(t)\right]\hat{j}}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{\cos(t + \Delta t) - \cos(t)}{\Delta t}\hat{i} + \frac{\sin(t + \Delta t) - \sin(t)}{\Delta t}\hat{j}$$

$$= \frac{d(\cos t)}{dt}\hat{i} + \frac{d(\sin t)}{dt}\hat{j} = -\sin(t)\hat{i} + \cos(t)\hat{j}$$

In other words (and more generally), if we have a function $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$,

then $\dfrac{d\vec{r}}{dt} = \dfrac{dx}{dt}\hat{i} + \dfrac{dy}{dt}\hat{j} + \dfrac{dz}{dt}\hat{k}$. We differentiate by simply differentiating each

component function, and when we do, we get another vector! Graphically, things

look like this.

$$\vec{r}'(t) = \frac{d\vec{r}}{dt} = \lim_{\Delta t \to 0} \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t}$$

As Δt gets smaller, our quotient $\dfrac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t}$ gets closer and closer to being

tangent to our curve. Thus, if we evaluate $\vec{r}'(t)$ at a particular value of t, then we get

a vector which is tangent to the curve $\vec{r}(t)$ at the point corresponding to that t value.

Now let's do a drawing.

We're going to start over with the function $\vec{r}(t) = 2\cos(t)\hat{i} + 2\sin(t)\hat{j}$, where $0 \le t \le 2\pi$.

This is also a circle with center at the origin, but this time the radius is 2. Also, it's

easy to see that $\dfrac{d\vec{r}}{dt} = -2\sin(t)\hat{i} + 2\cos(t)\hat{j}$. Furthermore, if we evaluate this derivative

at $t = \dfrac{\pi}{2}$, we get $\dfrac{d\vec{r}}{dt}\bigg|_{t=\pi/2} = -2\sin(\pi/2)\hat{i} + 2\cos(\pi/2)\hat{j} = -2\hat{i}$. The corresponding point

on our circle when $t = \dfrac{\pi}{2}$ is $(0,2)$. Thus, if we set $x = -2t$ and $y = 2$ with $0 \le t \le 1$,

then we can add the graph of this tangent vector to the graph of our vector-valued

function. And here it is!
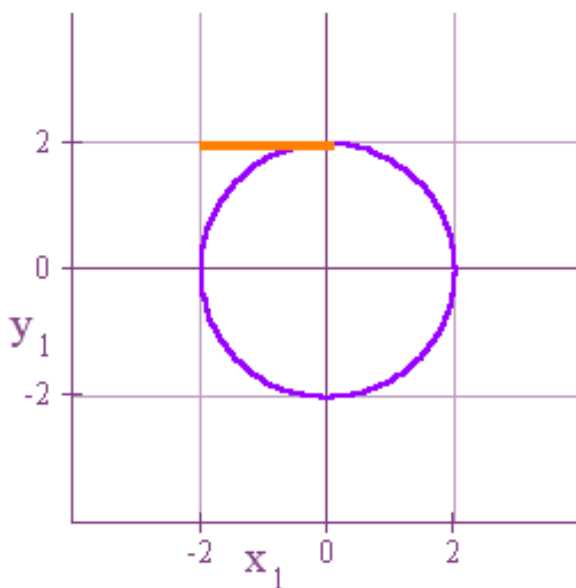
Out tangent vector points in the direction it does because with our particular

parametrization, we are tracing the circle in the counterclockwise direction. Also,

notice that the length of our tangent vector $\dfrac{d\vec{r}}{dt}\bigg|_{t=\pi/2} = -2\sin(\pi/2)\hat{i} + 2\cos(\pi/2)\hat{j} = -2\hat{i}$ is

2. However, for many applications that will come later, we will want to have not just

any ol' tangent vector, but a tangent vector of length 1, what we call the unit tangent

vector. To get a formula for the unit tangent, just divide the derivative of our vector-

valued function by its length. Thus, unit tangent $= T = \dfrac{\dfrac{d\vec{r}}{dt}}{\left\| \dfrac{d\vec{r}}{dt} \right\|} = \dfrac{\vec{r}'(t)}{\left\| \vec{r}'(t) \right\|}$. In the case

before us, $T = -\hat{i}$, and a parametrization for this starting at $(0,2)$ is $x = -t$, $y = 2$, and

$0 \le t \le 1$. And here's the new picture.
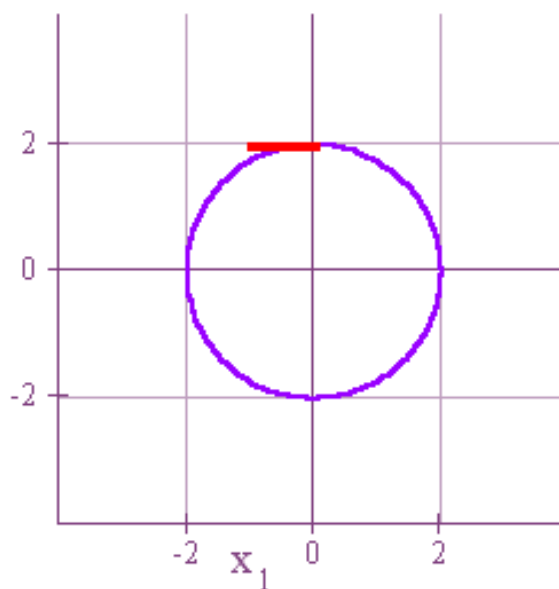
As you can see, then length has been adjusted from 2 to 1.

In addition to the unit tangent vector, we can also define a unit normal vector which is

perpendicular to the unit tangent. If we are in 3-dimensional space, the calculations

can become a little complicated because at any given point on a circle, there are an

infinite number of unit vectors that are perpendicular to the unit tangent. Just imagine

taking on such vector at a point and then rotating it full circle about the unit tangent,

and you'll see how we wind up with an infinite number. We have an infinite number

of directions such a normal vector could point in. However, in two dimensions things

are much simpler. For example, in the diagram above there are only two directions

the unit normal could point in when placed at the designated point. It could point

either straight up or straight down, and that's it! Consequently, things are simpler,

and once we've found a unit tangent vector, the procedure for getting the unit normal

will be very simple indeed. And here's how we do it. Suppose that at some point on

a curve in two dimensions that the unit tangent vector is $T = S\hat{i} + R\hat{j}$. Then the unit

normal vector is $N = R\hat{i} - S\hat{j}$. We just switch the components S and R, and then we

add a minus sign in front of the S. And how do we know that the result is a unit

vector? Because since T is a unit vector, it's length is $\|T\| = \sqrt{S^2 + R^2} = 1$, and thus,

$\|N\| = \sqrt{R^2 + (-S)^2} = 1$. And how do we know that N is perpendicular to T? Because

the dot product $T \cdot N = SR - RS = 0$. For our diagram above, $T = -\hat{i}$, and so $N = \hat{j}$.
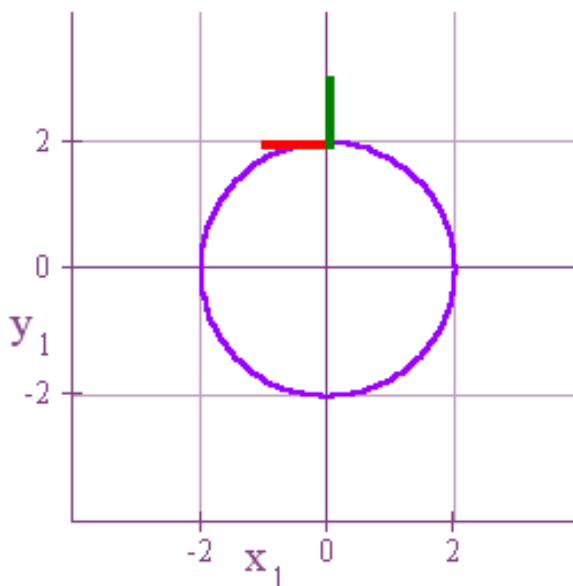
By the way, there is a simple way to understand the direction of both the unit tangent

and the unit normal vector. First, it should be clear that the unit tangent vector

basically points in the direction that corresponds to the direction in which you are

tracing out your curve. Because the circle above comes into being in a

counterclockwise direction as we let our angle increase from 0 to 2π , our unit

tangent takes its direction from that orientation. If we were tracing our curve in the

clockwise direction, then the unit tangent would likewise point in the opposite

direction. Once we understand the direction of our unit tangent, then it's very easy to

know which direction the unit normal will point in. Just imagine this. If you are

facing the direction in which the unit tangent is pointing, then the unit normal will

always be at right angles to it pointing to your right. This is a consequence of the

way in which we defined the unit tangent, and if you graph a few examples by hand on paper, then you'll understand why our unit normal is always the one that points to the right of the unit tangent.

Now let's repeat everything with another point on the circle. In particular, let's let $t = \dfrac{\pi}{4}$. Then,

$$\vec{r}(\pi/4) = 2\cos(\pi/4)\hat{i} + 2\sin(\pi/4)\hat{j} = \sqrt{2}\,\hat{i} + \sqrt{2}\,\hat{j}$$
$$\vec{r}'(\pi/4) = -2\sin(\pi/4)\hat{i} + 2\cos(\pi/4)\hat{j} = -\sqrt{2}\,\hat{i} + \sqrt{2}\,\hat{j}$$
$$\left\|\vec{r}'(\pi/4)\right\| = \sqrt{2+2} = \sqrt{4} = 2$$
$$T(\pi/4) = \frac{\vec{r}'(\pi/4)}{\left\|\vec{r}'(\pi/4)\right\|} = \frac{-1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}$$
$$N(\pi/4) = \frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}$$

Well that's all we need to know in order to set up parametric equations for graphing these vectors. For the unit tangent, we use,

$$x = \sqrt{2} - \frac{1}{\sqrt{2}}t$$
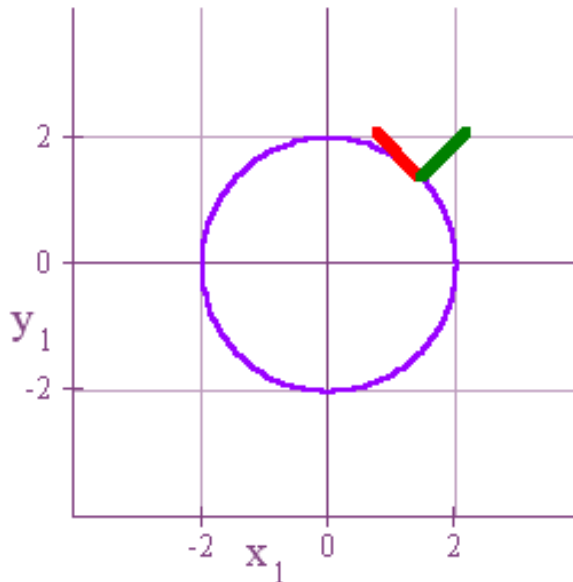$$y = \sqrt{2} + \frac{1}{\sqrt{2}}t$$
$$0 \le t \le 1$$

And for the unit normal we us,

$$x = \sqrt{2} + \frac{1}{\sqrt{2}}t$$

$$y = \sqrt{2} + \frac{1}{\sqrt{2}}t$$

$$0 \le t \le 1$$

And here's the result!

In many situations, it is common to think of our parameter t as representing time, and

$\vec{r}(t)$ as giving the position of an object at time t. In this setup, we already have a

word for the rate of change of position with respect to time. That's what we call

velocity. Thus, we will often designate the first derivative of our vector-valued

function $\vec{r}(t)$ as a velocity vector $\vec{v}(t)$. In other words, $\vec{r}'(t) = \dfrac{d\vec{r}}{dt} = \vec{v}(t) = velocity$.

Furthermore, we define speed as the length of our velocity vector,

$speed = \left\| \vec{r}\,'(t) \right\| = \left\| \dfrac{d\vec{r}}{dt} \right\| = \left\| \vec{v}(t) \right\|$. In ordinary language and even in some other

mathematics courses we often use the terms velocity and speed to mean the same

thing. However, once we reach this level, we make a distinction between the two.

Velocity is a vector quantity with both magnitude and direction, and speed is simply

the scalar length of our velocity vector. For our circle above,

$\vec{r}(t) = 2\cos(t)\hat{i} + 2\sin(t)\hat{j}$ and $\vec{v}(t) = -2\sin(t)\hat{i} + 2\cos(t)\hat{j}$. When $t = \dfrac{\pi}{4}$, this becomes

$\vec{v}(\pi/4) = -2\sin(\pi/4)\hat{i} + 2\cos(\pi/4)\hat{j} = -\sqrt{2}\,\hat{i} + \sqrt{2}\,\hat{j}$ and the corresponding speed is

$\left\| -\sqrt{2}\,\hat{i} + \sqrt{2}\,\hat{j} \right\| = \sqrt{2+2} = \sqrt{4} = 2$.

Another thing we might look at now is the rate of change of velocity with respect to

time, and we also already have a name for this. We call it acceleration. Thus,

$\vec{r}\,''(t) = \vec{v}\,'(t) = \dfrac{d\vec{v}}{dt} = \vec{a}(t) = acceleration$. For our circle example we have
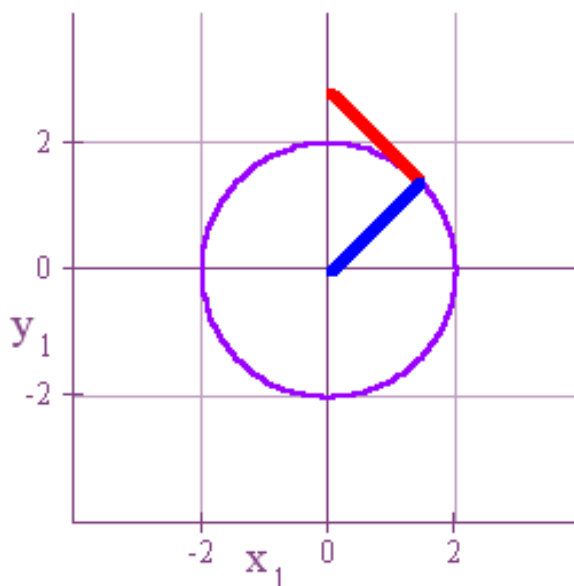
$\vec{a}(t) = \dfrac{d\vec{v}}{dt} = -2\cos(t)\hat{i} - 2\sin(t)\hat{j}$, and at $t = \dfrac{\pi}{4}$ we get

$\vec{a}(\pi/4) = -2\cos(\pi/4)\hat{i} - 2\sin(\pi/4)\hat{j} = -\sqrt{2}\,\hat{i} - \sqrt{2}\,\hat{j}$. The parametric equations we'll use

for graphing line segments for these vectors are as follows,

Velocity	Acceleration
$x = \sqrt{2} - \sqrt{2}t$	$x = \sqrt{2} - \sqrt{2}t$
$y = \sqrt{2} + \sqrt{2}t$	$y = \sqrt{2} - \sqrt{2}t$
$0 \le t \le 1$	$0 \le t \le 1$

And here is our graph.

We can see in this case that the acceleration is directly towards the center of the circle as we orbit it in the counterclockwise direction. By the way, there is also a name for the derivative of acceleration. It's called the jerk! That makes sense because if an acceleration is not smooth, then the ride feels jerky. However, we won't mess with any jerks in this course.

Once we understand differentiation of vector-valued functions, we can just as easily integrate them. In particular, since differentiation is done individually for each separate component, integration must be performed in a similar manner. Thus,

$$\int_a^b \vec{r}(t)\,dt = \int_a^b x(t)\,dt\,\hat{i} + \int_a^b y(t)\,dt\,\hat{j} + \int_a^b z(t)\,dt\,\hat{k}\,, \text{ and}$$

$$\int \vec{r}(t)\,dt = \int x(t)\,dt\,\hat{i} + \int y(t)\,dt\,\hat{j} + \int z(t)\,dt\,\hat{k} + \vec{C}$$

Notice that in the indefinite integrals you will pick up arbitrary constants of integration in each component. The sum of these arbitrary constants results in the constant vector $\vec{C}$ that we've added on at the end. Now let's take an example!

Let's suppose that our velocity vector is $\vec{v}(t) = -2\sin(t)\hat{i} + 2\cos(t)\hat{j}$ and that when $t = 0$, $\vec{r}(0) = 3\hat{i} + 2\hat{j}$. We now want to find $\vec{r}(t)$, and just as you have done in other courses, it all boils down to integrating one function and then using your condition to evaluate the constant. Thus, $\vec{r}(t) = \int \vec{v}(t)\,dt = 2\cos(t)\hat{i} + 2\sin(t)\hat{j} + \vec{C}$. Now using the given condition we get $\vec{r}(0) = 2\cos(0)\hat{i} + 2\sin(0)\hat{j} + \vec{C} = 2\hat{i} + \vec{C} = 3\hat{i} + 2\hat{j} \Rightarrow \vec{C} = \hat{i} + 2\hat{j}$.

Thus, $\vec{r}(t) = 2\cos(t)\hat{i} + 2\sin(t)\hat{j} + \left(\hat{i} + 2\hat{j}\right) = \left(2\cos(t) + 1\right)\hat{i} + \left(2\sin(t) + 2\right)\hat{j}$. Notice that the graph of our end result would be a circle of radius 2 that has had its center shifted from the origin exactly one unit to the right and 2 units up, i.e. to the point $(1,2)$.

Now let's think about another problem involving curves created by vector-valued functions. In particular, how do we find the length, also called the arc length, of our

curve as our parameter varies from *a* to *b*? Let's look at a diagram to get some

inspiration.
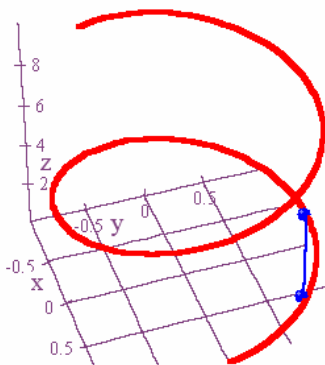
The curve above is the graph of $\vec{r}(t) = \cos(t)\hat{i} + \sin(t)\hat{j} + t\hat{k}$ with $0 \le t \le 10$. We have

two points marked on the curve in blue and a straight line is connecting the two

points. We now ask ourselves what the length of the arc is from the first point to the

second point. Well, if our points are close enough together, then we can effectively

approximate the true arc length by simply finding the straight line distance between

the points. And we know how to do the latter because our distance formula tells us

that the distance between the two points will be the square root of the sum of the

change in *x* squared plus the change in *y* squared plus the change in *z* squared. In

other words, $\text{distance} = \sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2}$. Now let's go back and take several points

off of our curve and approximate the arc lengths between them in the same way.
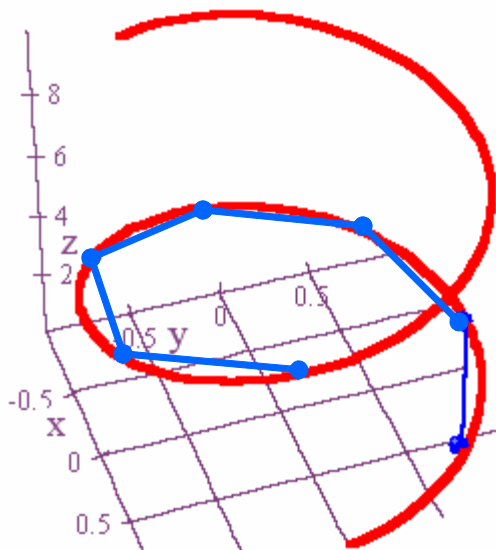
The result now is that for the curve overall, $Arc\,Length \approx \sum \sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2}$. Look

now at how we can rewrite this formula.

$$\sum \sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2} = \sum \sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2 + \left(\frac{\Delta z}{\Delta t}\right)^2} \cdot \Delta t$$

And finally, we can generate a formula for the exact arc length just by taking the limit

of this last expression as Δt goes to 0.

$$Arc\,Length = \lim_{\Delta t \to 0} \sum \sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2 + \left(\frac{\Delta z}{\Delta t}\right)^2} \cdot \Delta t$$

$$= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt = \int_a^b \|\vec{r}'(t)\| dt = \int_a^b \|\vec{v}(t)\| dt$$

This last bit is an unexpected surprise. What it tells us, though, is that if we have our curve defined parametrically by a vector-valued function, then the arc length of the curve as t varies from a to b is equal to the integral from a to b of the norm of the velocity function. It works out this way, of course, because if

$$\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k} \text{, then } \vec{v}(t) = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k} \text{, and}$$

$$\|\vec{v}(t)\| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} . \text{ Applying this to a circle of radius } r \text{ in two}$$

dimensions with center at the origin, we could take $\vec{r}(t) = r\cos(t)\hat{i} + r\sin(t)\hat{j}$ where $0 \le t \le 2\pi$. Thus,

$$x = r\cos(t) \quad \frac{dx}{dt} = -r\sin(t)$$

$$y = r\sin(t) \quad \frac{dy}{dt} = r\cos(t)$$

From this we get that $\sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2} = \sqrt{r^2\sin^2(t) + r^2\cos^2(t)} = \sqrt{r^2} = r$.

Consequently, the circumference of a circle is,

$$Arc\,Length = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^{2\pi} r\, dt = rt\Big|_0^{2\pi} = 2\pi r$$

Of course, you already knew that the circumference was 2π times the radius, but it's nice to know how to actually (and easily!) prove that fact.

Now let's look at something else that comes out of this formula for arc length. We can make the upper integration limit of our arc length integral a variable so that we can define a function that returns to us the arc length of our curve from a starting value a to a stopping value t. When we do this, we'll have to replace the variable t in the integrand by a dummy variable so that we're not using t in two different ways. Once we've done that, this is what we get.

$$s(t) = \int_a^t \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2}\, du$$

And now what is nice about this formula is that the Fundamental Theorem of Calculus tells us exactly how to differentiate it with respect to t. According to our Fundamental Theorem,

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} = \|\vec{r}'(t)\| = \|\vec{v}(t)\| = speed$$

Again, the fact that the derivative is equal to the speed might be totally unexpected, but when we think about it makes sense. If $s(t)$ is measuring the scalar distance traveled as a function of time, then of course the rate at which this distance changes with respect to time is nothing other than the speed at which we are traveling, i.e.

$$\frac{ds}{dt} = \|\vec{v}(t)\| = speed$$

Now let's look at one more calculus application, how to measure how curved a curve actually is. The definition that we are going to use for curvature is,

$$\kappa = \left\| \frac{dT}{ds} \right\|$$

The letter on the left is the Greek letter *kappa*, and the expression on the right is the norm of the rate of change of the unit tangent vector with respect to arc length. Let's think about this for a moment. As our unit tangent vector moves along a curve, then one thing that is not going to change is its length. In fact, the only thing that can change is its direction. If its direction changes a lot with regard to a small change in arc length, then we've got some curvature. On the other hand, if a small change in arc length produces only a small change in direction, then we have very little curvature. Thus, it makes perfectly good sense to measure curvature as the magnitude of the rate of change of the unit tangent vector with respect to arc length. Now, however, the only problem is how do we compute this? This is where the chain rule helps us! According to the chain rule, we should have that

$$\frac{dT}{dt} = \frac{dT}{ds} \cdot \frac{ds}{dt} = \frac{dT}{ds} \cdot \left\| \vec{r}'(t) \right\| = \frac{dT}{ds} \cdot \left\| \frac{d\vec{r}}{dt} \right\|$$

Hence, do a little division and you get,

$$\frac{dT}{ds} = \frac{\dfrac{dT}{dt}}{\left\| \dfrac{d\vec{r}}{dt} \right\|} \Rightarrow \kappa = \left\| \frac{dT}{ds} \right\| = \frac{\left\| \dfrac{dT}{dt} \right\|}{\left\| \dfrac{d\vec{r}}{dt} \right\|}$$

This gives us a formula we can use for computations! For example, if a parametrization for a circle of radius r with center at the origin is

$\vec{r}(t) = r\cos(t)\hat{i} + r\sin(t)\hat{j}$ with $0 \le t \le 2\pi$, then $\vec{r}'(t) = -r\sin(t)\hat{i} + r\cos(t)\hat{j}$, $\|r'(t)\| = r$,

$T = \dfrac{\vec{r}'(t)}{\|\vec{r}'(t)\|} = -\sin(t)\hat{i} + \cos(t)\hat{j}$, and $\dfrac{dT}{dt} = -\cos(t)\hat{i} - \sin(t)\hat{j}$. Thus,

$$\kappa = \left\|\frac{dT}{ds}\right\| = \frac{\left\|\dfrac{dT}{dt}\right\|}{\left\|\dfrac{d\vec{r}}{dt}\right\|} = \frac{1}{r}$$

In other words, at any point on a circle of radius r, the curvature is $\dfrac{1}{r}$. This result

helps us to better understand what curvature actually means. For example, if you

think of the earth as a big circle with a large radius, then you can understand why it

looks flat to you. With a radius of almost 4000 miles, the curvature is pretty darn

small.

There exist several equivalent formulas that can be developed for calculating

curvature, but this is as far as I'm going to go in this book. If you need an equivalent

formula, it's always there in other books. For right now, though, I think the original

formula is the best to focus on in order to better understand what curvature actually

means.

Now let's go back and see how much of what we've learned can be applied to the

formula that we derived for a cycloid generated by a circle of radius 1,

$$\vec{r}(t) = (t - \sin(t))\hat{i} + (1 - \cos(t))\hat{j}$$
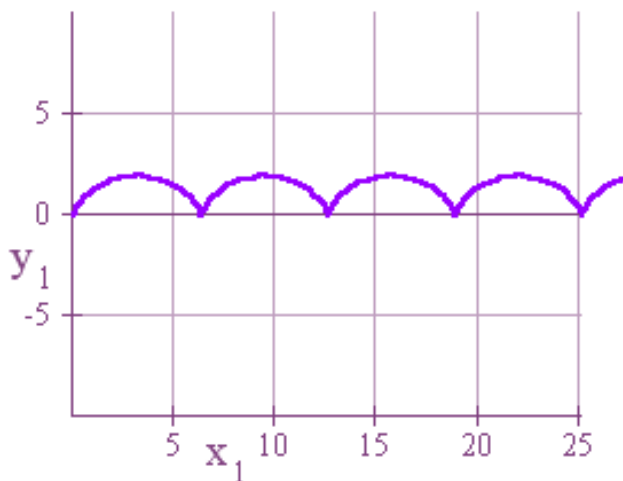
Here are some important derivations,

$$\vec{r}(t) = (t - \sin(t))\hat{i} + (1 - \cos(t))\hat{j}$$
$$\vec{r}'(t) = (1 - \cos(t))\hat{i} + \sin(t)\hat{j}$$
$$\|\vec{r}'(t)\| = \sqrt{1 - 2\cos(t) + \cos^2(t) + \sin^2(t)} = \sqrt{2 - 2\cos(t)}$$

From these equations it follows that,

$$T(t) = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|} = \frac{1 - \cos(t)}{\sqrt{2 - 2\cos(t)}}\hat{i} + \frac{\sin(t)}{\sqrt{2 - 2\cos(t)}}\hat{j}$$

$$N(t) = \frac{\sin(t)}{\sqrt{2 - 2\cos(t)}}\hat{i} - \frac{1 - \cos(t)}{\sqrt{2 - 2\cos(t)}}\hat{j}$$
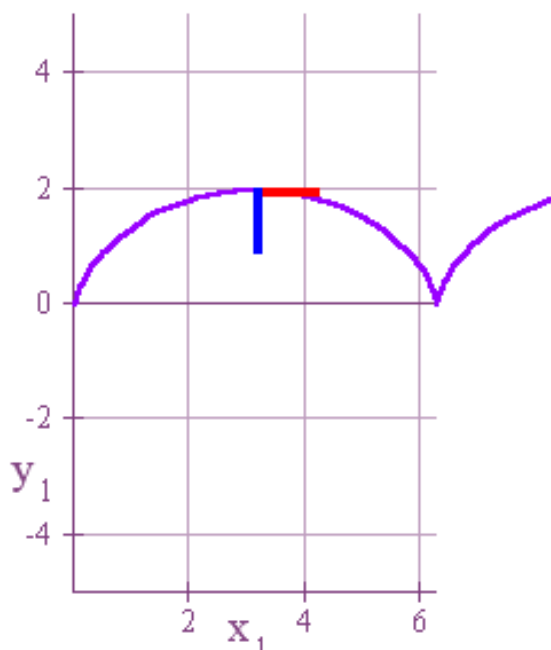
Hence, with a little (or a lot!) of effort we can obtain

$$T'(t) = \frac{1}{2}\frac{\sin(t)}{\sqrt{2-2\cos(t)}}\hat{i} - \frac{1}{2}\frac{1-\cos(t)}{\sqrt{2-2\cos(t)}}\hat{j}$$

$$\kappa(t) = \frac{\|T'(t)\|}{\|\vec{r}'(t)\|} = \frac{\sqrt{\dfrac{1}{4}\cdot\dfrac{\sin^2(t)}{2-2\cos(t)} + \dfrac{1}{4}\cdot\dfrac{1-2\cos t+\cos^2(t)}{2-2\cos(t)}}}{\sqrt{2-2\cos(t)}} = \frac{1}{2}\cdot\frac{1}{\sqrt{2-2\cos(t)}}$$

This means that when $t = \pi$, $\vec{r}(\pi) = \pi\hat{i} + 2\hat{j}$, $T(\pi) = \hat{i}$, and $N(\pi) = -\hat{j}$. Let's graph

these! The parametric equations we can use to graph $T(\pi) = 2\hat{i}$ and $N(\pi) = -2\hat{j}$ so

that they start at the point $(\pi, 2)$ are,

Unit Tangent

$x = \pi + t$

$y = 2$

$0 \le t \le 1$

Unit Normal

$x = \pi$

$y = 2 - t$

$0 \le t \le 1$

Notice, as we mentioned before, that if you are facing in the direction of the unit

tangent, then the unit normal is pointing to your right.

Now let's take a look at the curvature when $t = \pi$. We have that

$\kappa(\pi) = \dfrac{1}{2} \cdot \dfrac{1}{\sqrt{2 - 2\cos(\pi)}} = \dfrac{1}{4}$. That means that a circle of radius $\dfrac{1}{\kappa} = 4$ should fit into

the curve at that point quite nicely. The center of such a circle should be at $(\pi, -2)$,

four units below the point $(\pi, 2)$ when moving in the direction of the unit normal.

Hence, we can parametrize this circle by finding a parametrization for a circle of

radius 4 with center at the origin and then add the vector $\pi\hat{i} - 2\hat{j}$ to it. The end result

is $\vec{w}(t) = \left(4\cos(t) + \pi\right)\hat{i} + \left(4\sin(t) - 2\right)\hat{j}$. Now just set x equal to the first component, y

to the second component, and let $0 \le t \le 1$, and we've got our parametrization. Here's the final graph!
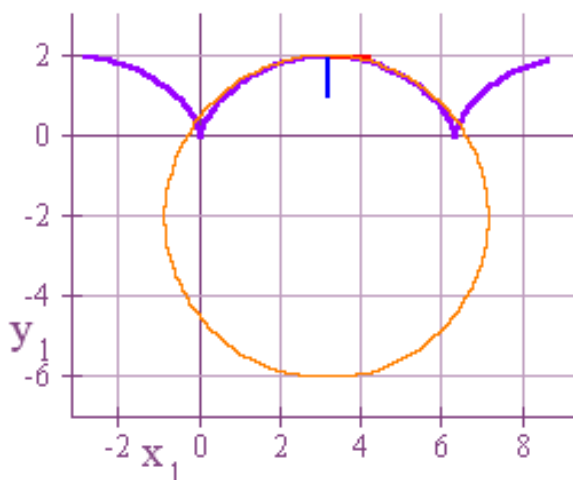
Close enough for government work!

Now let's see what we can accomplish with regard to arc length. To find the length of the arc from $t = 0$ to $t = 2\pi$, we're going to have to use some trigonometry identities, so let's get everything set up now. Recall that $\sin^2(t) = \dfrac{1 - \cos(2t)}{2}$. This implies that $\sin^2\left(\dfrac{t}{2}\right) = \dfrac{1 - \cos(t)}{2}$ which implies that $4\sin^2\left(\dfrac{t}{2}\right) = 2 - 2\cos(t)$. It's this last equation that we are going to have to use soon. We now have from our earlier

work that the length of the arc from $t = 0$ to $t = 2\pi$ is

$$s(2\pi) = \int_0^{2\pi} \|\vec{r}'(t)\| \, dt = \int_0^{2\pi} \sqrt{2 - 2\cos(t)} \, dt = \int_0^{2\pi} \sqrt{4\sin^2\left(\frac{t}{2}\right)} \, dt = \int_0^{2\pi} 2\sin\left(\frac{t}{2}\right) dt$$

$$= -4\cos\left(\frac{t}{2}\right)\Big|_0^{2\pi} = 4 + 4 = 8$$

And it's just that simple!

While we're at it, let's also find the area under the curve from $t = 0$ to $t = 2\pi$. You would think this would be easy, but since we have our curve defined in terms of a parameter t instead of the usual $y = f(x)$, this will be a little tricky. First, let's recall our parametric equations for this cycloid,

$$x = t - \sin(t)$$
$$y = 1 - \cos(t)$$

Observe, also, that if we did have y as a function of x, then the area under the curve would be given by $\int_{t=0}^{t=2\pi} y \, dx$. Now comes the clever part. If we compute $\dfrac{dx}{dt}$ above,

then we get $\dfrac{dx}{dt} = 1 - \cos(t) = y$. Hence,

$$\int_{t=0}^{t=2\pi} y \, dx = \int_0^{2\pi} y \frac{dx}{dt} \, dt = \int_0^{2\pi} (1 - \cos(t))((1 - \cos(t))) \, dt = \int_0^{2\pi} 1 - 2\cos(t) + \cos^2(t) \, dt$$

$$= \int_0^{2\pi} 1 - 2\cos(t) + \frac{1 + \cos(2t)}{2} \, dt = t - 2\sin(t) + \frac{t}{2} + \frac{\sin(2t)}{4}\Big|_0^{2\pi} = 3\pi$$

Now let's make an important definition and look at just one more thing. A smooth

curve, geometrically speaking, is a curve without any sharp points on the graph.

Notice that our cycloid is not smooth since we get a sharp point at intervals of 2π .

We also say that a curve is smooth on an interval if it has a smooth parametrization.

By this we mean that there exists a parametrization $\vec{r}(t)$ with the property that $\vec{r}'(t)$ is

continuous and $\vec{r}'(t) \neq \vec{0}$, the zero vector. It's not hard to see why we want $\vec{r}'(t)$ to be

continuous. After all, if $\vec{r}'(t)$ changed direction in an abrupt manner, then that could

result in a sharp point. But why do we want $\vec{r}'(t) \neq \vec{0}$. Well, first of all, notice that for

the cycloid where we have a sharp point at $t = 2\pi$,

$\vec{r}'(2\pi) = \left(1 - \cos(2\pi)\right)\hat{i} + \sin(2\pi)\hat{j} = \vec{0}$. Recall also that $\vec{r}'(t) = \vec{v}(t) = velocity$. If you are

walking along, then it's possible to let your velocity vary continuously and still have

a sharp point on your path. All you have to do is gradually let your speed decrease to

zero and then start up in a totally different direction. Thus, if we really don't want

any sharp points on our curve, then we also have to require that $\vec{v}(t) = \vec{r}'(t) \neq \vec{0}$. And

that's the truth!