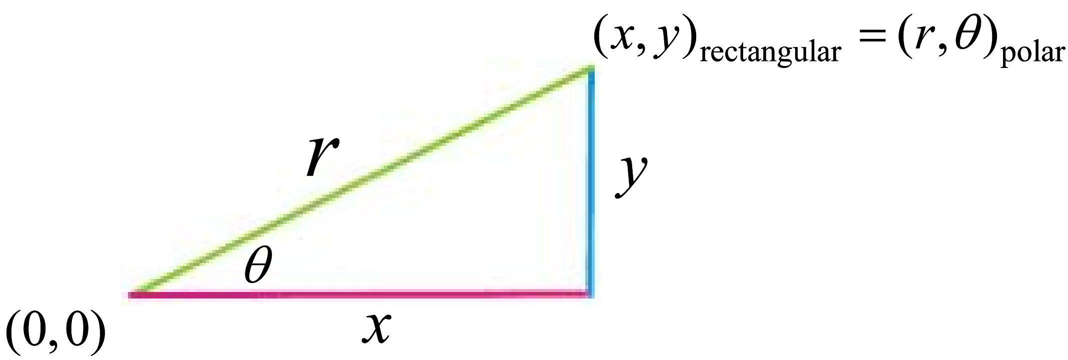
CHAPTER 2

CYLINDRICAL AND SPHERICAL COORDINATES

Okay! You've finished chapter 1, studied graphs of functions of several variables in exhaustive detail, and now you know everything you need to know about graphs in 3-dimensions. Right? Wrong!!! Remember polar coordinates? Well, we hope you remember polar coordinates. As you hopefully recall, polar coordinates are another way to locate points in 2-dimensional space, and they serve as an alternative to the usual rectangular coordinates that locate a point using values off of the x and y axes. In polar coordinates, instead of using x and y to locate a point, we specify the point's distance from the x-axis and the angle that it makes with the positive x-axis. Here's the diagram that explains it all.

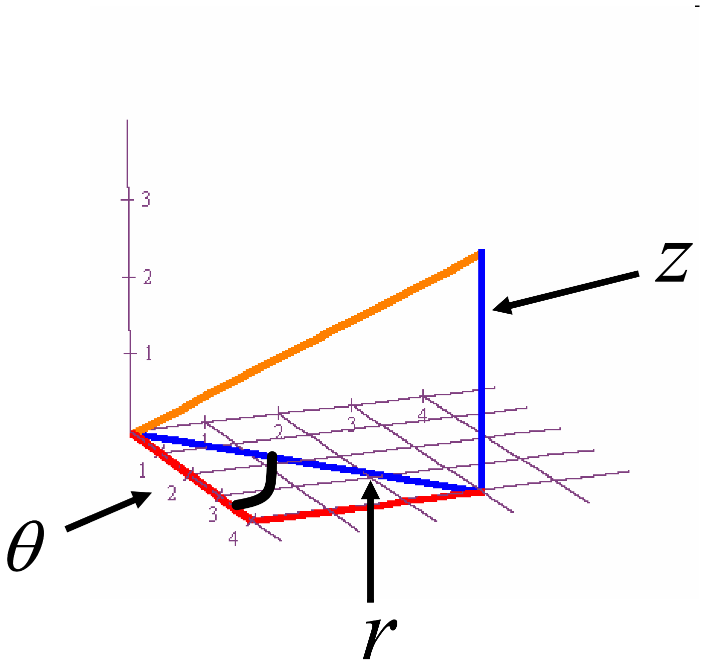


From trigonometry, we also know that the following relationships will exist between rectangular coordinates (x, y) and polar coordinates (r,θ):

$x = r\cos\theta$, $y = r\sin\theta$, $x^2 + y^2 = r^2$, and $slope = \tan\theta = y/x$.

The cylindrical and spherical coordinate systems that we are going to look at now might be thought of as extensions of polar coordinates into higher dimensions. We'll begin with cylindrical coordinates.

In cylindrical coordinates for a point in 3-dimensional space, we basically make the first two coordinates polar and then keep the elevation z that we are above or below the xy-plane as our third coordinate. Here's an illustration that shows how we can locate a point in space with coordinates (r, θ, z).



The position of the point as it might lie in the xy-plane is determined by the polar coordinates r and θ, and then the distance above or below this plane is determined by the z coordinate. The important relationships between rectangular coordinates, (x, y, z), and cylindrical coordinates, (r, θ, z), are as follows:

$x = r\cos\theta$, $y = r\sin\theta$, $z = z$, $x^2 + y^2 = r^2$, and $\tan\theta = y/x$, where we usually also have that $0 \le \theta \le 2\pi$, $0 \le r < \infty$, and $-\infty < z < \infty$.

At his point, there are only a few things we really want to know when it comes to cylindrical coordinates. We want to understand a few basic graphs and when it might be easier to approach a problem using cylindrical rather than rectangular coordinates, and we want to be able to easily convert from one coordinate system to the other. Well start with some conversion examples.

<u>Example 1:</u> Convert $(2, \pi/2, 3)$ from cylindrical, (r, θ, z), to rectangular, (x, y, z), coordinates.

This is the kind of conversion that's easiest to do. We just apply directly the formulas $x = r\cos\theta$ and $y = r\sin\theta$. Thus, $(2, \pi/2, 3)_{\text{cylindrical}}$ is equal to

$\left(2\cos(\pi/2), 2\sin(\pi/2), 3\right) = (0, 2, 3)_{\text{rectangular}}$.

<u>Example 2:</u> Convert $(2,\pi/4,1)$ from cylindrical, (r,θ,z), to rectangular, (x,y,z), coordinates.

Again, this is just $\left(2\cos(\pi/4),2\sin(\pi/4),1\right)=\left(2/\sqrt{2},2/\sqrt{2},1\right)=\left(\sqrt{2},\sqrt{2},1\right)_{\text{rectangular}}$.

<u>Example 3:</u> Convert $\left(1,\sqrt{3},4\right)$ from rectangular, (x,y,z), to cylindrical, (r,θ,z), coordinates.

Going from rectangular to cylindrical is only slightly more difficult than going from cylindrical to rectangular. In particular, we will have to use the arctangent function to help us find our angle, and if necessary, we will have to make adjustments to our result to ensure that the angle lies between 0 and 2π. We will also need to calculate the value for r from the values we have for x and y. In this example, $\tan\theta=\frac{y}{x}=\frac{\sqrt{3}}{1}$, and so $\theta=\tan^{-1}\sqrt{3}=60°=\frac{\pi}{3}$. Also, $r=\sqrt{x^2+y^2}=\sqrt{1^2+\left(\sqrt{3}\right)^2}=\sqrt{4}=2$. Therefore, $\left(1,\sqrt{3},4\right)_{\text{rectangular}}=\left(2,\frac{\pi}{3},4\right)_{\text{cylindrical}}$.

<u>Example 4:</u> Convert $\left(-\sqrt{3},1,2\right)$ from rectangular, $\left(x,y,z\right)$, to cylindrical, $\left(r,\theta,z\right)$, coordinates.

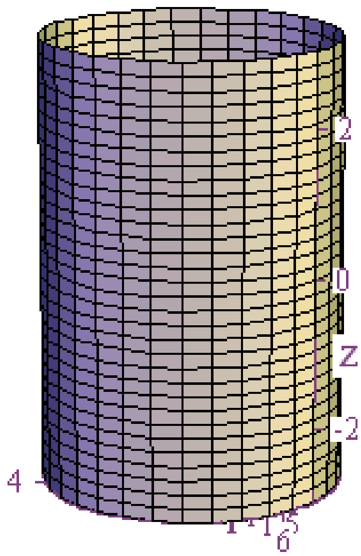
This is similar to the previous example except that when we try to find our angle, we have to make an adjustment to get it into the second quadrant where our *x* and *y* coordinates lie. Thus, we know that $\tan\theta=\frac{y}{x}=\frac{1}{-\sqrt{3}}$ and $\tan^{-1}\left(\frac{1}{-\sqrt{3}}\right)=-30°=\frac{-\pi}{6}$. But as you can see, our angle puts us in the wrong quadrant with respect to *x* and *y*, and so we are going to have to find the related second quadrant angle. We can do this by either adding $180°$ to $-30°$ or π radians to $\frac{-\pi}{6}$. We'll chose the latter format since it is more customary to express this angle in radians. Also,

$r=\sqrt{x^2+y^2}=\sqrt{(-\sqrt{3})^2+1^2}=\sqrt{4}=2$, and therefore,

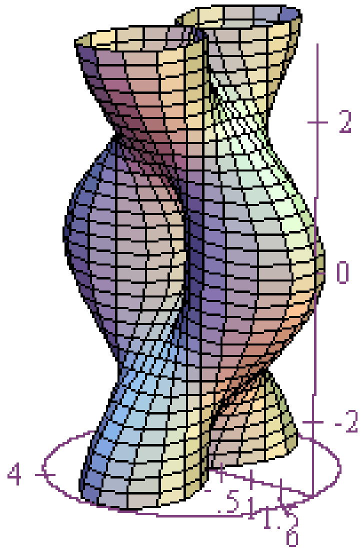
$$\left(-\sqrt{3},1,2\right)_{\text{rectangular}}=\left(2,\frac{5\pi}{6},2\right)_{\text{cylindrical}}.$$

Now let's look at a few graphs in cylindrical coordinates. These are often set up as $r=f(\theta,z)$ where the dependent or output variable is *r* and the two independent or input variables are θ and *z*. It doesn't have to always be set up this way, but nonetheless, that's how it is often done and how we will do it in the few examples below.

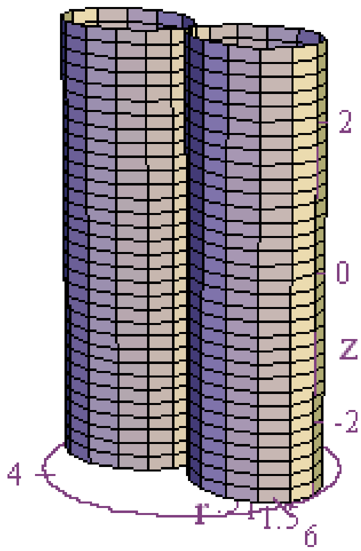
The type of surface that is often described using cylindrical coordinates is one in which the points in the domain are related to either a disk or a circle of radius r. For example, suppose we set $r = 2$, let θ vary so that $0 \le \theta \le 2\pi$, and let z vary so that $-3 \le z \le 3$. Then the result is a cylindrical surface about the z-axis of radius 2 and extending from $z = -3$ to $z = 3$. Because we can describe this surface so easily using cylindrical coordinates, that's where the particular name comes from.



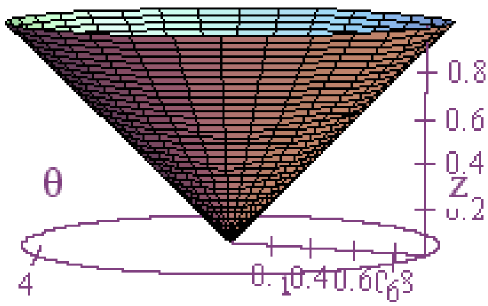
<u>Example 5:</u> There are a variety of interesting surfaces that may be generated using cylindrical coordinates, but they are often more difficult to analyze than surfaces in rectangular coordinates. One of my favorite surfaces, though, is $r = 1 + \cos 2\theta \cos z$ where $0 \le \theta \le 2\pi$ and $-3 \le z \le 3$. I like to think of this graph as two twin souls inextricably joined together.



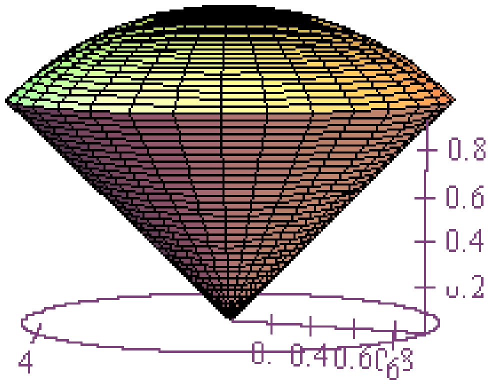
You can see in this graph how multiplying the main term by $\cos z$ creates a nice wave as we ascend up the z-axis. Without this multiplier, the graph $r = 1 + \cos(2\theta)$ looks much more mundane.



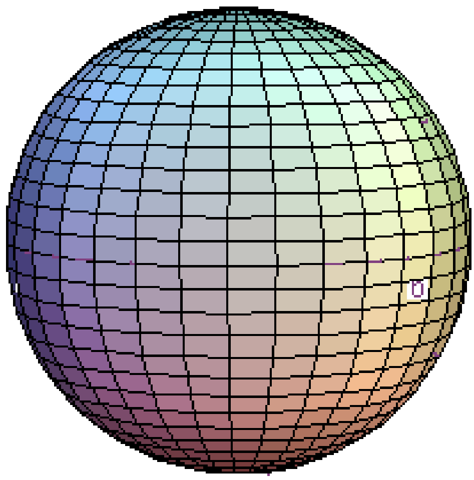
Example 6: We can also use cylindrical coordinates to easily create an ice cream cone and then fill it with ice cream. Recall that we did a representation of a cone earlier by graphing $z = \sqrt{x^2 + y^2}$. If we change this equation to cylindrical coordinates, then it becomes $z = \sqrt{x^2 + y^2} = \sqrt{r^2} = r$. Thus, if we set $0 \le \theta \le 2\pi$, $0 \le r \le 1$, and $z = r$, then we get a cone whose sides have length $\sqrt{2}$.



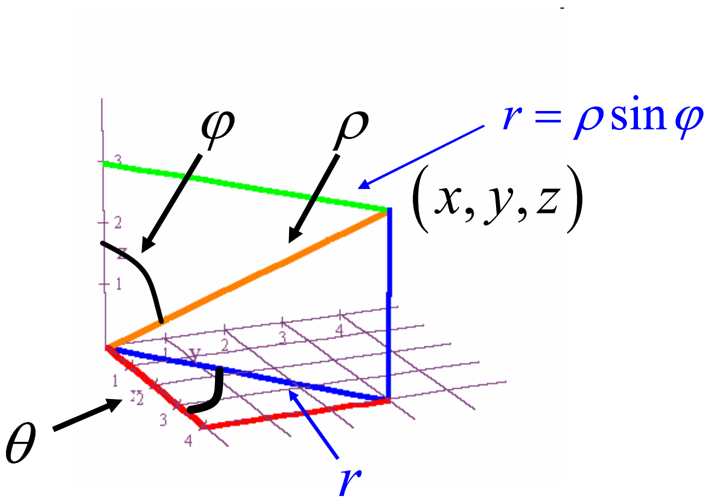
If we want to now fill the ice cream cone with ice cream, then we can generate a portion of a sphere of radius $\sqrt{2}$ for this purpose. The usual equation in rectangular coordinates for a sphere of this radius with center at the origin is $x^2 + y^2 + z^2 = 2$. We can change this to cylindrical coordinates as $z = \sqrt{2-(x^2+y^2)} = \sqrt{2-r^2}$ where $0 \le r \le 1$ and $0 \le \theta \le 2\pi$. And this is the result when we combine this surface with our cone.



Now let's start looking at spherical coordinates. To specify a point in spherical coordinates, we need three things – the distance ρ the point is from the origin, the angle θ the point would make with the positive *x*-axis if we repositioned it in the *xy*-plane, and the angle φ that the point makes with the positive *z*-axis. And as you might expect, this is the easiest way to represent a sphere. For the sphere of radius $\sqrt{2}$ the equation simply becomes $\rho = \sqrt{2}$ where $0 \le \theta \le 2\pi$ and $0 \le \varphi \le \pi$. The result is below.



Before we do any more graphs, however, let's look at how we might convert coordinates from one system to the other. Study this diagram well!



From this diagram we can deduce the following relationships:

$x = r\cos(\theta) = \rho\sin(\varphi)\cos(\theta)$, $y = r\sin(\theta) = \rho\sin(\varphi)\sin(\theta)$, and $z = \rho\cos(\varphi)$. We can also conclude that $\rho^2 = x^2 + y^2 + z^2$, $\tan\theta = \dfrac{y}{x}$, and $\varphi = \arccos\dfrac{z}{\sqrt{x^2+y^2+z^2}} = \dfrac{z}{\rho}$.

Furthermore, we generally assume that $0 \le \rho < \infty$, $0 \le \varphi \le \pi$, and $0 \le \theta \le 2\pi$. Now let's look at some examples involving conversions of coordinates.

Example 7: Just as with cylindrical coordinates, it's much easier to convert from spherical coordinates, (ρ,θ,φ), to rectangular coordinates, (x,y,z), than it is the other way around. For example, suppose we want to convert $(1,0,0)_{\text{spherical}}$ to rectangular. We just do the calculations $x = \rho\sin\varphi\cos\theta = 1\cdot\sin 0\cdot\cos 0 = 0$, $y = \rho\sin\varphi\sin\theta = 1\cdot\sin 0\cdot\sin 0 = 0$, and $z = \rho\cos\varphi = 1\cdot\cos 0 = 1$. Hence, $(1,0,0)_{\text{spherical}} \to (0,0,1)_{\text{rectangular}}$.

Example 8: Let's do another one! This time we'll convert $(2,\pi/2,3\pi/4)_{\text{spherical}}$ to rectangular coordinates. Once again,

$x = \rho\sin\varphi\cos\theta = 2\cdot\sin 3\pi/4\cdot\cos\pi/2 = 2\cdot\left(\sqrt{2}/2\right)\cdot 0 = 0$,

$y = \rho\sin\varphi\sin\theta = 2\cdot\sin 3\pi/4\cdot\sin\pi/2 = 2\cdot\left(\sqrt{2}/2\right)\cdot 1 = \sqrt{2}$, and

$z = \rho\cos\varphi = 2\cdot\cos 3\pi/4 = 2\cdot\left(-\sqrt{2}/2\right) = -\sqrt{2}$. Hence,

$(2,\pi/2,3\pi/4)_{\text{spherical}} \to (0,\sqrt{2},-\sqrt{2})_{\text{rectangular}}$.

<u>Example 9:</u> Now let's go in the opposite direction and convert $(-3,0,0)_{\text{rectangular}}$ to spherical coordinates. First, $\rho=\sqrt{(-3)^2+0^2+0^2}=3$. Also, if we graph this point, then it's position in the *xy*-plane with respect to the positive *x*-axis tells us that $\theta=\pi$, and its position with respect to the positive *z*-axis tells us that $\varphi=\dfrac{\pi}{2}$. Therefore,

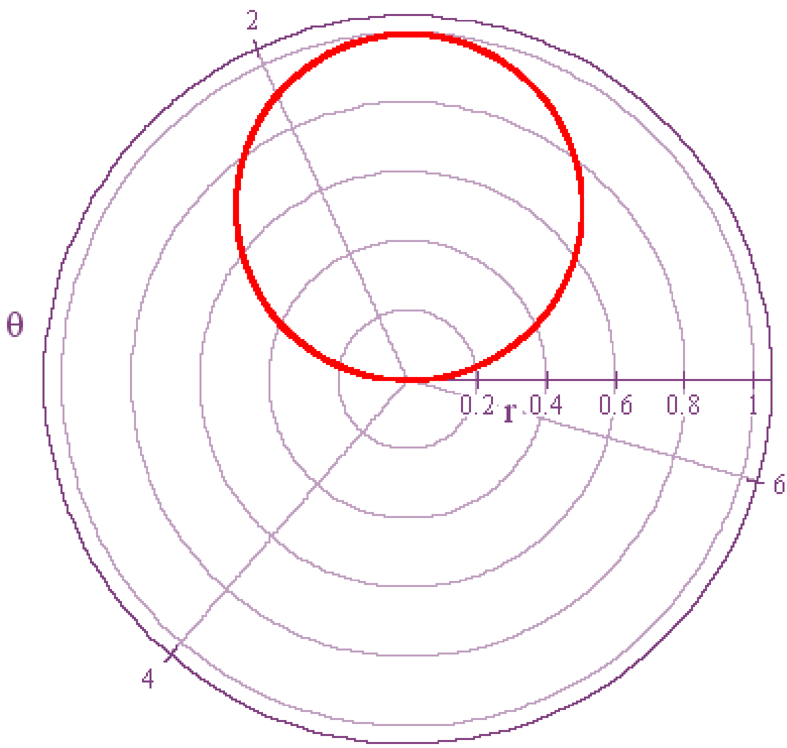
$$(-3,0,0)_{\text{rectangular}} \to \left(3,\pi,\frac{\pi}{2}\right)_{\text{spherical}}.$$

<u>Example 10:</u> For our last conversion, let's convert $(1,-1,-\sqrt{2})_{\text{rectangular}}$ to spherical coordinates. We have $\rho=\sqrt{1^2+(-1)^2+(-\sqrt{2})^2}=\sqrt{4}=2$. Also, since $\tan^{-1}\dfrac{y}{x}=\tan^{-1}\dfrac{-1}{1}=\tan^{-1}(-1)=\dfrac{-\pi}{4}$, our angle θ has to be related $\dfrac{-\pi}{4}$. However, considering the quadrant our point is in with respect to the *xy*-plane and given that we want our angle to be between 0 and 2π, we conclude that $\theta=\dfrac{7\pi}{4}$. Our calculation for φ is a little more straight forward. We have $\varphi=\cos^{-1}\left(\dfrac{z}{\rho}\right)=\cos^{-1}\left(\dfrac{-\sqrt{2}}{2}\right)=\dfrac{3\pi}{4}$.

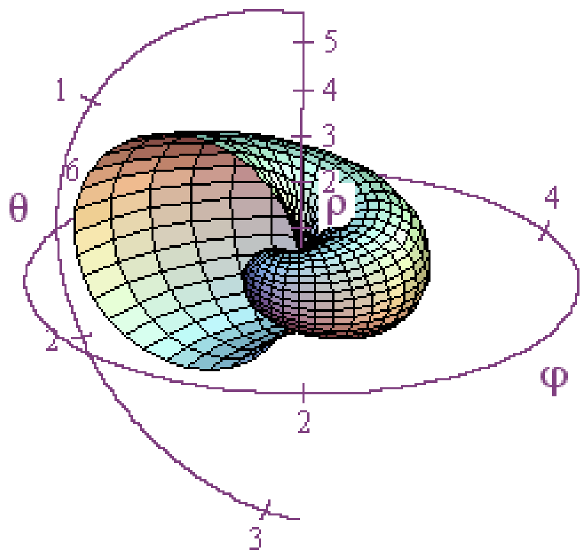
Hence, $(1,-1,-\sqrt{2})_{\text{rectangular}} \to \left(2,\dfrac{7\pi}{4},\dfrac{3\pi}{4}\right)_{\text{spherical}}$.

<u>Example 11:</u> Now that we've done a few conversions, let's look at some graphs. As we've mentioned, just as it's easiest to describe a cylinder in cylindrical coordinates,

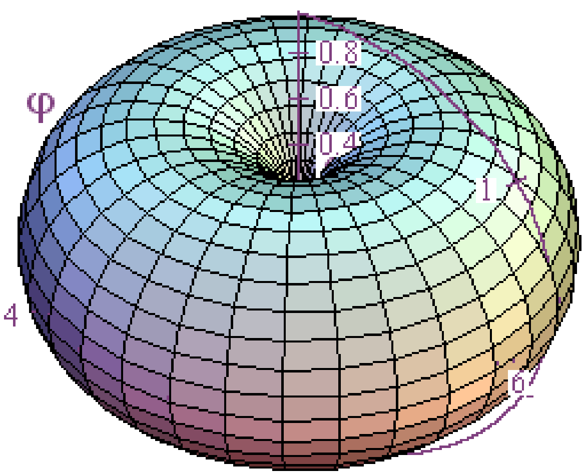
so are spherical coordinates the easiest way to describe a sphere. But of course, we can generate many other kinds of graphs, too, using spherical coordinates. However, they are often more difficult to analyze than graphs in good ol' rectangular *xyz*-coordinates. Nonetheless, let's think about $\rho = \sin\varphi \cdot \left(1.3^{\theta}\right)$. When $\theta = 0$, this becomes $\rho = \sin\varphi$, and in polar coordinates the graph of $r = \sin\theta$ is a circle of radius 1 that passes through the origin.



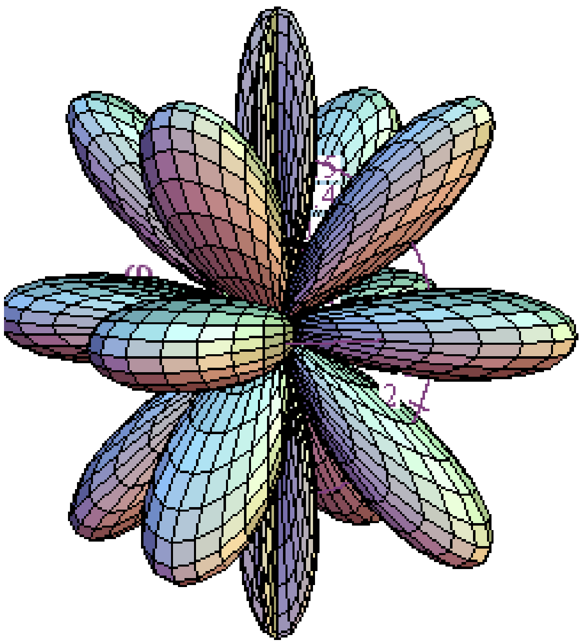
Thus, when $\theta = 0$ we expect the graph of $\rho = \sin\varphi \cdot \left(1.3^{\theta}\right)$ to be just a circle. However, as θ increases, ρ will also increase, and the radius of our circle gets larger. The end result is as follows.



If, on the other hand, we just graphed $\rho = \sin\varphi$ in three dimensions, then the radius of our circle would remain equal to one, and the following graph would result.



Example 12: For our next example, let's consider $\rho = \cos 4\varphi \cdot \cos 2\theta$. The first factor should cause oscillations in the graph as φ varies from 0 to π, and the second factor should create oscillations as we let θ go from 0 to 2π. The end result, as you can see below, is pretty ugly!

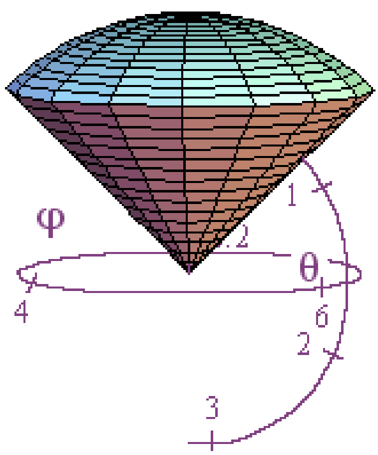


Example 13: And for our final example, let's see how we can create our filled ice cream cone in spherical coordinates. In this instance, it is particularly easy. The top part of our previous ice cream cone was a sphere of radius $\sqrt{2}$, and it extended downward $45°$ from the top. That means we want to let φ vary from 0 to $\pi/4$, and then let θ vary form 0 to 2π. And finally, for the cone part, we fix φ at $\pi/4$, let θ

vary form 0 to 2π, and let ρ vary from 0 to $\sqrt{2}$. A more succinct description of the two pieces is as follows.

Ice Cream Top	Cone
$\rho = \sqrt{2}$	$0 \le \rho \le \sqrt{2}$
$0 \le \theta \le 2\pi$	$0 \le \theta \le 2\pi$
$0 \le \varphi \le \frac{\pi}{4}$	$\varphi = \frac{\pi}{4}$

And the end result is the tasty graph below!



Finally, for future reference, here is a table that summarizes the relationships we talked about earlier between rectangular and cylindrical and spherical coordinates.

Cylindrical	Spherical
$x = r\cos\theta$	$x = r\cos(\theta) = \rho\sin(\varphi)\cos(\theta)$
$y = r\sin\theta$	$y = r\sin(\theta) = \rho\sin(\varphi)\sin(\theta)$
$z = z$	$z = \rho\cos(\varphi)$
$x^2 + y^2 = r^2$	$\rho^2 = x^2 + y^2 + z^2$
$\tan\theta = y/x$	$\tan\theta = \frac{y}{x}$
	$\varphi = \arccos\frac{z}{\sqrt{x^2 + y^2 + z^2}} = \frac{z}{\rho}$

Warning! Danger, Will Robinson, danger! Notation for spherical coordinates is not fixed! Some write (ρ,θ,φ), some write (ρ,φ,θ), some substitute r for ρ and switch θ and φ, and some do things that no one wants to know about! Always understand what convention your author is following before proceeding.