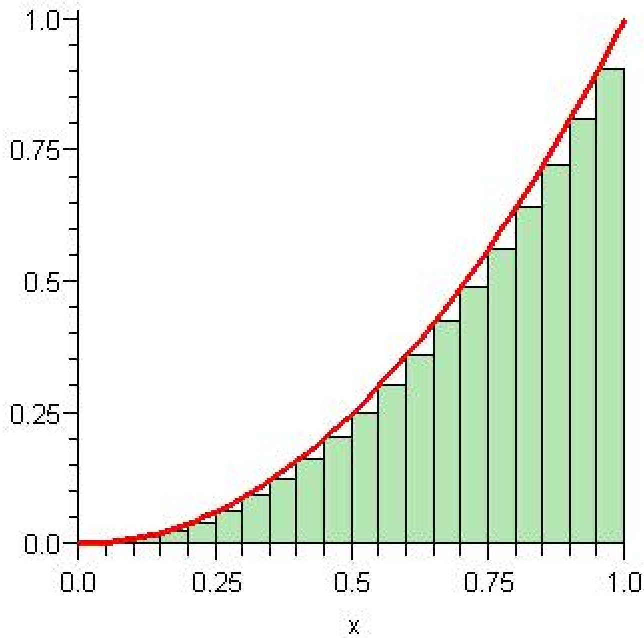
CHAPTER 10

LINE INTEGRALS

I'm sure everyone has fond memories of integrating a function like $f(x) = x^2$ from $x = 0$ to $x = 1$ and getting $\frac{1}{3}$ as the exact area under the curve.

$$\int_0^1 x^2\,dx = \left.\frac{x^3}{3}\right|_0^1 = \frac{1^3}{3} - \frac{0^3}{3} = \frac{1}{3}$$



What you didn't think about, probably, was that the interval you were integrating over was just a straight line, and that it might actually be possible to define integrals with respect to more arbitrary curves in space in a manner involving limits that's similar to what you did in previous calculus courses. And that's exactly what we're going to do! And we're going to call the result a "line integral."

Definition: If $f(x,y)$ is defined on a smooth curve C that is parametrized by $x = x(t)$ and $y = y(t)$, (or equivalently, $\vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath}$), where $a \le t \le b$, and if s represents arc length, then the line integral of $f(x,y)$ along C from the point $\left(x(a), y(a)\right)$ to the point $\left(x(b), y(b)\right)$ is,

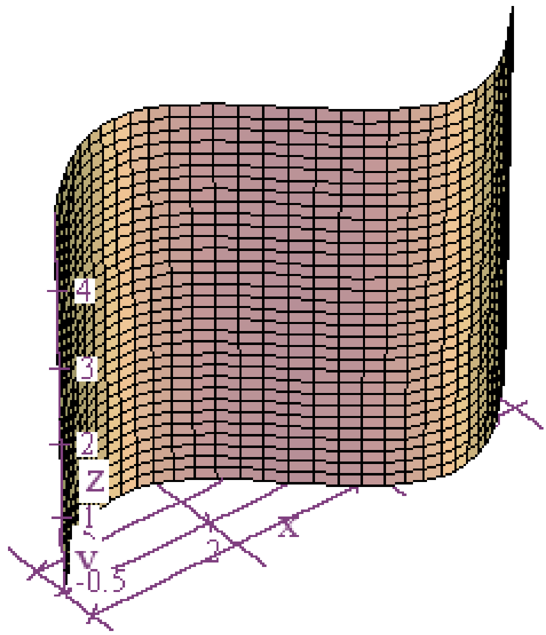
$$\int_C f(x,y)\,ds = \lim_{\Delta s \to 0} \sum f(x,y)\,\Delta s \text{ , provided this limit exists.}$$

Suppose we have something like a curtain whose bottom is curled along a wavy path, and suppose $f(x,y)$ represents the height of the curtain at particular point. Then the line integral defined above would give us back the area of this curtain in the same way that integrals you've studied in the past have represented area.

Let's now talk about how we evaluate line integrals. In practice, we almost always use our parametrization for our curve to turn our line integral into an ordinary integral like the ones you dealt with in previous calculus courses. Using our parametrization, we get,

$$\int_C f(x,y)\,ds = \int_a^b f\left(x(t),y(t)\right)\cdot\frac{ds}{dt}\,dt = \int_a^b f\left(x(t),y(t)\right)\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt\,.$$

Remember how a long time ago we talked about arc length $s = s(t)$ for a vector-valued function $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}$, and we saw that $\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}$? Well, here's where we are going to use that formula. To now make things a little more specific, let's suppose we have a curtain that is 5 feet high, but it's furled a bit so that it's base describes a sine curve going from 0 to 2π feet. What is the area of the curtain? In this case, just take $f(x,y)=5$ and $\vec{r}(t) = t\,\hat{i} + \sin(t)\,\hat{j}$ where $0 \le t \le 2\pi$. Then our curtain will to look something like this,



And to find the area, just evaluate the line integral.

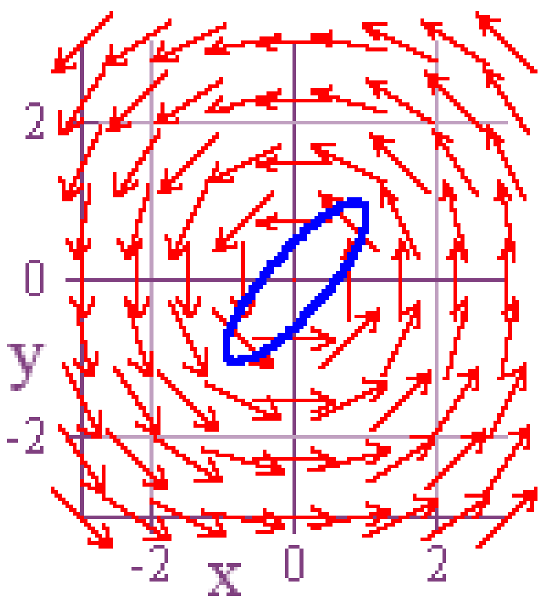
$$\int_C f(x,y)\,ds = \int_0^{2\pi} 5\cdot\frac{ds}{dt}\,dt = 5\int_0^{2\pi}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt = 5\int_0^{2\pi}\sqrt{1+\cos^2(t)}\,dt \approx 8.88\,ft^2$$

What's probably not obvious to you at this point is that we can integrate our function, if we wish, just with respect to x or solely with respect to y by using an appropriate parametrization. Here are the formulas for setting up those integrals.

$$\int_C f(x,y)\,dx = \int_a^b f\big(x(t),y(t)\big)\cdot\frac{dx}{dt}\,dt$$

$$\int_C f(x,y)\,dy = \int_a^b f\big(x(t),y(t)\big)\cdot\frac{dy}{dt}\,dt$$

These integrals with respect to x and y, far from being a mere mathematical abstraction, will actually be very important to us in the practical applications that follow. In particular, one of the most important applications of line integrals is to compute the work done by a vector field in pushing a particle along a curve. Thus, let's suppose we have a vector field $\vec{F}(x,y)$ and a smooth curve C that is parametrized in such a way that the curve is traced in the counterclockwise direction as t goes from $t=a$ to $t=b$. If the initial point of C coincides with the terminal point, then we'll also call C a closed curve, and if C doesn't intersect itself, except at the endpoints, then we'll call it simple. In general, we'll say that any simple, closed curve traced in a counterclockwise direction is positively oriented and any simple, closed curve traced in the clockwise direction is negatively oriented. Now recall that $\text{work} = \text{force} \times \text{distance}$.



Recall also that if our displacement is represented by a vector $\vec{D}$ and if the object displaced is acted upon by a force $\vec{F}$ pointing in a different direction, then the work done is equal to the component of $\vec{F}$ in the direction of the distance vector $\vec{D}$ times the length of $\vec{D}$. This gives us the following formula that we've seen before:

$$Work = comp_{\vec{D}}(\vec{F}) \bullet \left\|\vec{D}\right\| = \left\|\vec{F}\right\|\cos(\theta) \bullet \left\|\vec{D}\right\| = \left\|\vec{F}\right\|\left\|\vec{D}\right\|\cos(\theta) = \vec{F} \bullet \vec{D}$$

If our curve C is smooth and if the displacement of our particle is small, then as a result of local linearity, our displacement vector at a point is approximately equal to the change in arc length times the corresponding unit tangent vector. Hence,

$$Work \approx F \bullet \left(\Delta s \cdot T\right) = \left(F \bullet T\right) \cdot \Delta s$$

If we partition our curve C into a series of subintervals of length Δs, then the total work done by the force field in moving the particle along the curve C is:

$$Work \approx \sum \left(F \bullet T\right) \cdot \Delta s$$

$$\Rightarrow Work = \lim_{\Delta s \to 0} \sum \left(F \bullet T\right) \cdot \Delta s = \int_C F \bullet T \, ds$$

There are many different ways in which we can write this last formula. In particular,

$$\int_C F \bullet T\, ds = \int_C (F \bullet T)\frac{ds}{dt}dt = \int_C (F \bullet T)\|r'(t)\|dt = \int_C \left(F \bullet \frac{r'(t)}{\|r'(t)\|} \right)\|r'(t)\|dt$$

$$\int_C \left(F \bullet \frac{r'(t)}{\|r'(t)\|} \cdot \|r'(t)\| \right) dt = \int_C (F \bullet r'(t))\, dt = \int_C \left(F \bullet \frac{dr}{dt} \right) dt = \int_C F \bullet dr$$

This last expression, $\int_C \vec{F} \bullet d\vec{r}$, is how the line integral for work is often expressed.

Another formulation may be derived by writing $\vec{F}(x,y) = P(x,y)\hat{i} + Q(x,y)\hat{j}$ and $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}$, where $a \le t \le b$. This gives us,

$$\int_C F \bullet T\, ds = \int_C F \bullet dr = \int_C \left(F \bullet \frac{dr}{dt} \right) dt$$

$$= \int_a^b (P\hat{i} + Q\hat{j}) \bullet \left(\frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} \right) dt = \int_a^b \left(P\frac{dx}{dt} + Q\frac{dy}{dt} \right) dt = \int_C Pdx + Qdy$$

Now you can see why we pointed out that you can integrate line integrals just with respect to *x* or with respect to *y*. We will often want to write our work integral as $\int_C Pdx + Qdy$, and then use a parametrization for our curve to convert it into a more familiar integral that we know how to compute. For example, suppose our curve *C* is the unit circle parametrized in the counterclockwise direction by $\vec{r}(t) = \cos(t)\hat{i} + \sin(t)\hat{j}$ with $0 \le t \le 2\pi$ and that our vector field is $\vec{F}(x,y) = -y\hat{i} + x\hat{j}$.

Then the work done by this vector field in moving a particle around this curve is,

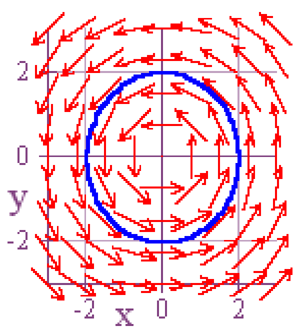
$$Work = \int_C \vec{F} \bullet d\vec{r} = \int_C P dx + Q dy = \int_a^b \left(-y \frac{dx}{dt} + x \frac{dy}{dt} \right) dt$$

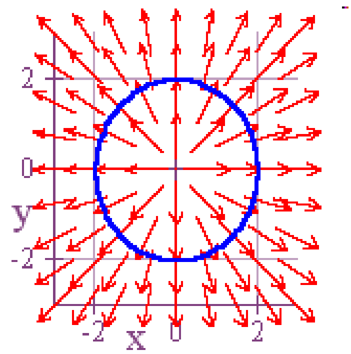
$$= \int_0^{2\pi} \left(-\sin(t)\left(-\sin(t)\right) + \cos(t)\cos(t) \right) dt = \int_0^{2\pi} \left(\sin^2 t + \cos^2 t \right) dt$$

$$\int_0^{2\pi} 1 dt = t \Big|_0^{2\pi} = 2\pi$$

Now let's go back for a minute to the formula $Work = \lim_{\Delta s \to 0} \sum \left(F \bullet T \right) \cdot \Delta s = \int_C F \bullet T \, ds$,

and let's contemplate it with respect to the images below.





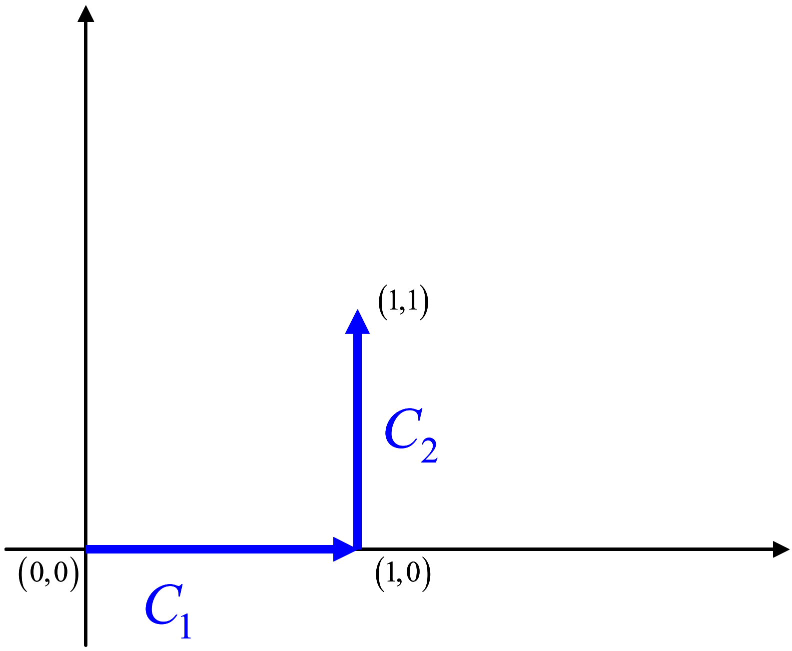
The dot product $\vec{F} \bullet T$ gives us the tangential component of the force when evaluated at a point on the circle above. In the first diagram, this tangential component is going to be quite a bit while in the second diagram, the tangential component is nil. Thus, for the first diagram the computation $\sum \left(F \bullet T \right) \cdot \Delta s$ will be much larger than what we would expect for the second. Also, when the tangential component of the force is substantial, then our vector field will tend to cause circulation about a curve as in the first diagram above, and when the tangential component of the force is completely lacking, then there will be no circulation created. The bottom line is that since $\lim_{\Delta s \to 0} \sum (F \bullet T) \cdot \Delta s = \int_C F \bullet T \, ds$, the same integral that measures the work done by a vector field along a given path also measures the circulation created by that vector field. In other words, $Circulation = \int_C F \bullet T \, ds = \int_C F \bullet dr = Work$.

Let's now look at a few more examples of line integrals. We'll include line integrals that involve circulation and work as well as more general line integrals. First, let's let our curve C be the circle of radius r with center at the origin parametrized by $\vec{r}(t) = r\cos(t)\hat{i} + r\sin(t)\hat{j}$ with $0 \le t \le 2\pi$. If we now evaluate $\int_C ds$, the integral along this curve with respect to arc length, then the result will simply be the length of our curve, and in particular, we will have found a simple way to derive the formula for the circumference of a circle.

$$\int_C ds = \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt = \int_0^{2\pi} \sqrt{(-r\sin t)^2 + (r\cos t)^2} \, dt = \int_0^{2\pi} \sqrt{r^2\sin^2 t + r^2\cos^2 t} \, dt$$

$$= \int_0^{2\pi} \sqrt{r^2\left(\sin^2 t + \cos^2 t\right)} \, dt = \int_0^{2\pi} \sqrt{r^2} \, dt = \int_0^{2\pi} r \, dt = rt\Big|_0^{2\pi} = 2\pi r$$

Now let's do a work integral where $\vec{F}(x,y) = x\hat{i} + y\hat{j}$, and our path is $C = C_1 + C_2$, the combination of the two straight line paths in the diagram below.



We can parametrize C_1 by $\vec{r}(t) = t\hat{i} + 0\hat{j} = t\hat{i}$ with $0 \le t \le 1$. Hence,

$$work_1 = \int_{C_1} \vec{F} \bullet d\vec{r} = \int_{C_1} Pdx + Qdy = \int_0^1 P\frac{dx}{dt}dt + Q\frac{dy}{dt}dt = \int_0^1 (t)(1) + (0)(0)\,dt = \left.\frac{t^2}{2}\right|_0^1 = \frac{1}{2}.$$

Similarly, we can parametrize C_2 by $\vec{r}(t) = 0\hat{i} + t\hat{j} = t\hat{j}$ with $0 \le t \le 1$. Thus,

$$work_2 = \int_{C_2} \vec{F} \bullet d\vec{r} = \int_{C_2} Pdx + Qdy = \int_0^1 P\frac{dx}{dt}dt + Q\frac{dy}{dt}dt = \int_0^1 (0)(0) + (t)(1)\,dt = \left.\frac{t^2}{2}\right|_0^1 = \frac{1}{2}$$

Therefore, the total work done is $work = \int_{C_1+C_2} \vec{F} \bullet d\vec{r} = \int_{C_1} \vec{F} \bullet d\vec{r} + \int_{C_2} \vec{F} \bullet d\vec{r} = \frac{1}{2} + \frac{1}{2} = 1$.

An important thing to realize in this example is that if we traversed our path C_2 in the opposite direction, then in our parametrization we could begin with $t=1$ and end with $t=0$, and this would result in the work along this path being equal to $-\frac{1}{2}$. Also, we can write this result as $\int_{-C_2} \vec{F}\bullet d\vec{r} = -\int_{C_2} \vec{F}\bullet d\vec{r}$. In other words, if you are doing a line integral and you reverse the direction in which you traverse your path, then all that this does is to change the sign of your original integral.

We now want to introduce a result for line integrals that is analogous to the Fundamental Theorem of Calculus, but before we do, we need to make some definitions. Remember when we said that you could always create a vector field by taking the gradient of a function of the sort $z = f(x,y)$? We'll it turns out that these kinds of vectors, i.e. gradient fields, are pretty special.

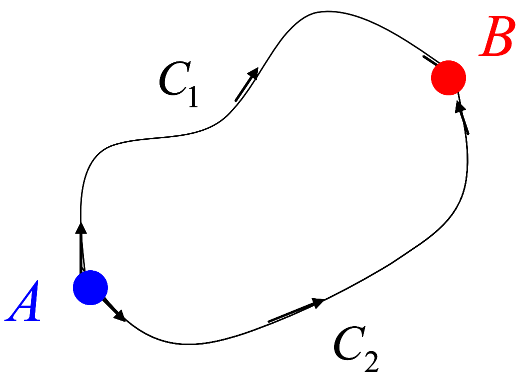
DEFINITION: A vector field $\vec{F} = P\hat{i} + Q\hat{j}$ is conservative if $\vec{F} = \nabla f$ for some function $z = f(x,y)$. In this case, f is called the potential function for $\vec{F}$.

DEFINITION: The line integral $\int_C \vec{F}\bullet d\vec{r}$ is independent of path if $\int_{C_1} \vec{F}\bullet d\vec{r} = \int_{C_2} \vec{F}\bullet d\vec{r}$ for any two paths C_1 and C_2 that have the same initial and terminal points.

Independence of path is going to mean that the value of the line integral depends only on the starting and stopping points of our curve. This should remind you of the Fundamental Theorem of Calculus where, when we have an antiderivative $F(x)$ of a continuous function $f(x)$, the value of the integral also depends only on the endpoints of our interval, $\int_a^b f(x)\,dx = F(b) - F(a)$. Our next step, though, is to give a condition that is equivalent to being independent of path.

THEOREM: A line integral $\int_{C_1} \vec{F} \bullet d\vec{r}$ is independent of path in *D* if and only if $\int_C \vec{F} \bullet d\vec{r} = 0$ for every closed path C in *D*.

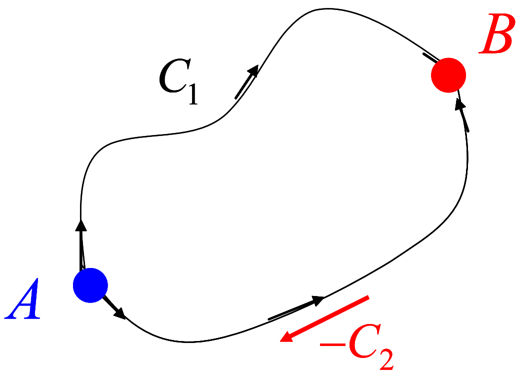
This actually pretty easy to see. Just consider the diagram below where we're trying to go from point *A* to point *B*.



If we have independence of path working for us, then $\int_{C_1} \vec{F} \bullet d\vec{r} = \int_{C_2} \vec{F} \bullet d\vec{r}$. However, if we reverse the direction along C_2, then $C = C_1 - C_2$ is a closed path, and

$\int_C \vec{F} \bullet d\vec{r} = \int_{C_1 - C_2} \vec{F} \bullet d\vec{r} = \int_{C_1} \vec{F} \bullet d\vec{r} + \int_{-C_2} \vec{F} \bullet d\vec{r} = \int_{C_1} \vec{F} \bullet d\vec{r} - \int_{C_2} \vec{F} \bullet d\vec{r} = 0$. You can just as easily start with the assumption that an integral around any closed path C is equal to zero, decompose this path into $C = C_1 - C_2$, and conclude that $\int_{C_1} \vec{F} \bullet d\vec{r} = \int_{C_2} \vec{F} \bullet d\vec{r}$.

Case closed!



Now let's prove the Fundamental Theorem of Line Integrals.

The Fundamental Theorem of Lines Integrals: Let C be a smooth curve with a smooth parametrization $\vec{r}(t) = x(t)\hat{i} + y(t)j$ for $a \le t \le b$, and let $z = f(x, y)$ be a function whose gradient ∇f is continuous on C. Then,

$$\int_C \nabla f \bullet d\vec{r} = f\left(x(b), y(b)\right) - f\left(x(a), y(a)\right).$$

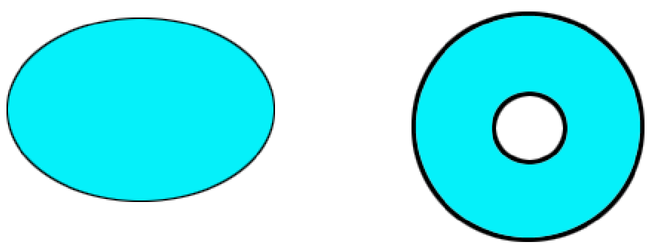
PROOF: $\int_C \nabla f \bullet d\vec{r} = \int_a^b \left(\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} \right) \bullet \frac{d\vec{r}}{dt} dt = \int_a^b \left(\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} \right) \bullet \left(\frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} \right) dt$

$= \int_a^b \left(\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \right) = \int_a^b \frac{df}{dt} dt = f\left(x(b), y(b)\right) - f\left(x(a), y(a)\right).$

Well, that proof was short and sweet! What it is telling us, however, is that if our vector field is a gradient field, then the value of our line integral depends only upon the end points. We can do something analogous to the Fundamental Theorem of Calculus, i.e. find a type of antiderivative and evaluate. Notice, too, how, once again, the chain rule has been used to complete the proof. What we want do now, though, is figure out some conditions that will let us know that our line integral is independent of path. First, however, we need another definition.

DEFINITION: A region R is connected if any two points in R can be joined by a path C that lies in R. A connected region R is simply connected if it contains no holes.

Below are two connected regions, but only the first is simply connected.



And now, here are some things that are equivalent to being independent of path.

THEOREM: Let $\vec{F} = P\hat{i} + Q\hat{j}$ be a vector field defined on an open simply connected region R, let C be a smooth curve in R, and suppose P and Q have continuous first order derivatives in R. Then the following are equivalent.

1. $\vec{F} = P\hat{i} + Q\hat{j}$ is conservative.
2. $\vec{F} = \nabla f$ for some function $z = f(x, y)$.
3. $\int_C \vec{F} \bullet d\vec{r}$ is independent of path.
4. $\int_C \vec{F} \bullet d\vec{r} = 0$ for every closed curve C in R.
5. $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$.
6. $curl\,\vec{F} = \nabla \times \vec{F} = \vec{0}$

The first four conditions are pretty obvious based upon what we've done so far. To see how number five works, suppose that $\vec{F} = P\hat{i} + Q\hat{j} = \nabla f$ for some function $z = f(x, y)$. Then $P = \dfrac{\partial f}{\partial x}$ and $Q = \dfrac{\partial f}{\partial y}$, and $\dfrac{\partial P}{\partial y} = \dfrac{\partial^2 f}{\partial y \partial x}$ and $\dfrac{\partial Q}{\partial x} = \dfrac{\partial^2 f}{\partial x \partial y}$. However, under the conditions above, these second order mixed partials are equal, and therefore, $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$. If we now rewrite this last equation as $\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y} = 0$, then this is just the statement that the scalar curl of $\vec{F}$ is zero. In other words, $curl\,\vec{F} = \nabla \times \vec{F} = \vec{0}$.

Finally, let's look at several examples to see how we can put this fundamental theorem into practice.

<u>Example 1:</u> $\vec{F} = 2x\hat{i} + 2y\hat{j}$ and C is any path from $(1,1)$ to $(2,2)$. Find $\int_C \vec{F} \bullet d\vec{r}$.

<u>Solution:</u> In this case, if $f(x,y) = x^2 + y^2$, then $\nabla f = \vec{F}$. Thus, the integral is independent of path and $\int_C \vec{F} \bullet d\vec{r} = f(2,2) - f(1,1) = 8 - 2 = 6$.

<u>Example 2:</u> If $\vec{F} = (3+2xy)\hat{i} + (x^2 - 3y^2)\hat{j}$, show that $\int_C \vec{F} \bullet d\vec{r}$ is independent of path.

<u>Solution:</u> Let $P = 3 + 2xy$ and $Q = x^2 - 3y^2$. Then $\dfrac{\partial P}{\partial y} = 2x = \dfrac{\partial Q}{\partial x}$. Therefore, $F = \nabla f$ for some function $z = f(x,y)$, and $\int_C \vec{F} \bullet d\vec{r}$ is independent of path.

<u>Example 3:</u> If $\vec{F} = (3+2xy)\hat{i} + (x^2 - 3y^2)\hat{j}$, find a potential function $z = f(x,y)$.

<u>Solution:</u> Let $P = 3 + 2xy$ and $Q = x^2 - 3y^2$. Then $\int P\,dx = 3x + x^2 y + g(y).$

Differentiate this result with respect to y and you get $x^2 + g'(y)$. Comparing this result with $Q = x^2 - 3y^2$, we see that we want $g'(y) = -3y^2$. An antiderivative for this with respect to y is $-y^3$. Hence, it suffices to let $f(x,y) = 3x + x^2 y - y^3$.

Example 4: If $\vec{F} = (3+2xy)\hat{i} + (x^2 - 3y^2)\hat{j}$, find $\int_C \vec{F} \bullet d\vec{r}$ where the curve C is defined by $\vec{r}(t) = e^t \sin(t)\hat{i} + e^t \cos(t)\hat{j}$ where $0 \le t \le \pi$.

Solution: On this curve, $x = e^t \sin(t)$ and $y = e^t \cos(t)$. Also, the integral is independent of path, and a potential function for $\vec{F}$ is $f(x,y) = 3x + x^2 y - y^3$. Hence,

$$\int_C \vec{F} \bullet d\vec{r} = f\left(x(\pi), y(\pi)\right) - f\left(x(0), y(0)\right)$$

$$= f(e^\pi \sin\pi, e^\pi \cos\pi) - f(e^0 \sin(0), e^0 \cos(0))$$

$$= f(0, -e^\pi) - f(0,1)$$

$$= (3 \cdot 0 + 0^2(-e^\pi) - (-e^\pi)^3) - (3 \cdot 0 + 0^2(1) - 1^3)$$

$$= e^{3\pi} + 1.$$

Congratulations! If you've made it this far, then you've got potential!!!