STOKES' THEOREM IN THREE DIMENSIONS - ANSWERS

In each problem below you are given a surface *S*, defined by $z = f(x,y)$, over a region *R*, defined by the given limits on *x* and *y*. Let C_R be the boundary of the region *R*, oriented counterclockwise, and let *C* be the corresponding boundary on the surface *S*, also oriented counterclockwise. Then if *F* is a vector field and *N* is the upward pointing unit normal vector for the surface *S*, use the higher dimensional version of Stokes' Theorem, $\int_C \vec{F} \bullet d\vec{r} = \iint_S (curl\, F \bullet N)\, dS$, to measure the circulation around the curve *C* that is caused by the vector field *F*.

Consider $z = f(x,y)$ as a level surface for a function $g = -f(x,y) + z$. Then $N = \dfrac{\nabla g}{\|\nabla g\|}$ will be an upward pointing unit normal to the surface $z = f(x,y)$, and we will have

$$\int_C \vec{F} \bullet d\vec{r} = \iint_S (curl\, F \bullet N)\, dS = \iint_R \left[(\nabla \times F) \bullet \frac{\nabla g}{\|\nabla g\|} \right] \cdot \|\nabla g\|\, dA = \iint_R \left[(\nabla \times F) \bullet \nabla g \right] dA.$$

1. $$\begin{aligned} &S: z = -x^2 - y^2 + 4 \\ &R: 0 \le x \le 1,\ 0 \le y \le 2 \\ &F = z\hat{i} + x\hat{j} + y\hat{k} \end{aligned}$$

$$g = x^2 + y^2 + z$$

$$\nabla g = 2x\hat{i} + 2y\hat{j} + \hat{k}$$

$$\nabla \times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ z & x & y \end{vmatrix} = \hat{i} + \hat{j} + \hat{k}$$

$$\int_C \vec{F} \bullet d\vec{r} = \iint_R \left[(\nabla \times F) \bullet \nabla g \right] dA = \int_0^1 \int_0^2 \left[\left(\hat{i} + \hat{j} + \hat{k} \right) \bullet \left(2x\hat{i} + 2y\hat{j} + \hat{k} \right) \right] dy\,dx$$

$$= \int_0^1 \int_0^2 (2x + 2y + 1)\, dy\,dx = \int_0^1 2xy + y^2 + y \Big|_0^2 dx = \int_0^1 (4x + 6)\, dx$$

$$= 2x^2 + 6x \Big|_0^1 = 2 + 6 = 8$$

$S: z = y$

2. $R: 0 \le x \le 1,\ 0 \le y \le 1 - x$

$F = -3y^2\,\hat{i} + 4z\,\hat{j} + 6x\,\hat{k}$

$g = -y + z$

$\nabla g = -\hat{j} + \hat{k}$

$$\nabla \times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -3y^2 & 4z & 6x \end{vmatrix} = -4\hat{i} - 6\,\hat{j} + 6y\,\hat{k}$$

$$\int_C \vec{F} \bullet d\vec{r} = \iint_R \left[\left(\nabla \times F \right) \bullet \nabla g \right] dA = \int_0^1 \int_0^{1-x} \left[\left(-4\hat{i} - 6\,\hat{j} + 6y\,\hat{k} \right) \bullet \left(-\hat{j} + \hat{k} \right) \right] dydx$$

$$= \int_0^1 \int_0^{1-x} \left(6 + 6y \right) dydx = \int_0^1 6y + 3y^2 \Big|_0^{1-x} dx = \int_0^1 \left[6(1-x) + 3(1-x)^2 \right] dx$$

$$= \int_0^1 (6 - 6x + 3 - 6x + 3x^2)\, dx = \int_0^1 (3x^2 - 12x + 9)\, dx$$

$$= x^3 - 6x^2 + 9x \Big|_0^1 = 1 - 6 + 9 = 4$$

$S: z = x^2 - y^2$

3. $R: -1 \le x \le 1,\ -\sqrt{1 - x^2} \le y \le \sqrt{1 - x^2}$

$F = x^2\,\hat{i} + y^2\,\hat{j} + z^2\,\hat{k}$

$g = -x^2 + y^2 + z$

$\nabla g = -2x\,\hat{i} + 2y\,\hat{j} + \hat{k}$

$$\nabla \times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 & y^2 & z^2 \end{vmatrix} = 0\,\hat{i} + 0\,\hat{j} + 0\,\hat{k} = \vec{0}$$

$$\int_C \vec{F} \bullet d\vec{r} = \iint_R \left[\left(\nabla \times F \right) \bullet \nabla g \right] dA = \iint_R \left[\left(0\,\hat{i} + 0\,\hat{j} + 0\,\hat{k} \right) \bullet \left(-2x\,\hat{i} + 2y\,\hat{j} + \hat{k} \right) \right] dA$$

$$= \iint_R 0\, dA = 0$$