DIVERGENCE THEOREM IN THREE DIMENSIONS - ANSWERS

In each problem below you are given a volume *V* bounded by a surface *S* along with a vector field *F*. If *N* is an outward pointing unit normal vector, then the outward flux across the surface created by the vector field *F* is, by the Divergence (Gauss') Theorem, equal to $\iint_S \vec{F} \bullet N dS = \iiint_V div\, \vec{F}\, dV = \iiint_V \nabla \bullet F\, dV$. Evaluate this integral for each problem below

1. V is the solid ball with surface S defned by $x^2 + y^2 + z^2 = 1$
   $F = x\hat{i} + y\,\hat{j} + z\,\hat{k}$

$$\iint_S \vec{F} \bullet N dS = \iiint_V \nabla \bullet F\, dV = \iiint_V (1+1+1)\, dV = 3\iiint_V dV = 3 \cdot \frac{4\pi}{3} = 4\pi$$

2. V is the cube defned by $0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1$, and S is the surface of the cube
   $F = x^2\,\hat{i} + y\,\hat{j} + z\,\hat{k}$

$$\iint_S \vec{F} \bullet N dS = \iiint_V \nabla \bullet F\, dV = \int_0^1\int_0^1\int_0^1 (2x+2)\, dxdydz = \int_0^1\int_0^1 x^2 + 2x\Big|_0^1 dydz$$

$$= \int_0^1\int_0^1 3\, dydx = \int_0^1 3y\Big|_0^1 dz = \int_0^1 3\, dz = 3z\Big|_0^1 = 3$$

3. V is the cylinder defned by $-1 \le x \le 1, -\sqrt{1-x^2} \le y \le \sqrt{1-x^2}, 0 \le z \le 4,$ and S is the surface of the cylinder
   $F = y\,\hat{i} - x\,\hat{j}$

$$\iint_S \vec{F} \bullet N dS = \iiint_V \nabla \bullet F\, dV = \iiint_V 0\, dV = 0$$