Lesson 17

MORE ON QUOTIENT GROUPS

In this lesson we want to give another couple of examples of quotient groups, and this time we will begin with an infinite group, the integers under the operation of addition. Symbolically, we denote the integers as $\mathbb{Z} = \{\ldots -3, -2, -1, 0, 1, 2, 3, \ldots\}$, and if our operation is addition, then the identity element is the number 0 and an inverse is just the opposite of a number. Thus, for example, the inverse of 2 is -2, the inverse of 3 is -3, and so on. Furthermore, since this group is abelian, every subgroup is normal.

One particular subgroup we want to look at is the set of all even integers. This is often denoted by $2\mathbb{Z} = \{\ldots -6, -4, -2, 0, 2, 4, 6, \ldots\}$. This set is closed under addition since the sum of two even integers is an even integer, and every element of this set has an inverse in this set since, for instance, if 2 is even, then so is -2. Thus, $2\mathbb{Z}$ is a normal subgroup of $\mathbb{Z}$.

If *G* is a group and *H* is a normal subgroup of *G*, then the quotient group of *G* by *H* is generally denoted by *G/H*. Thus, let's now think about what the quotient group $\mathbb{Z}/2\mathbb{Z}$ will look like. Clearly, the identity element is going to be $2\mathbb{Z}$, the set of all even integers. Also, if we add an even number to every even integer in this set, then we just get back the same set of even numbers. Hence, for example, $2\mathbb{Z} + 2 = 2\mathbb{Z}$. On the other hand, if we add an odd number to every even integer in $2\mathbb{Z}$, then we'll get back the entire set of odd integers. Consequently, $2\mathbb{Z} + 1 = \{\ldots -5, -3, -1, 1, 3, 5, \ldots\}$. Since the union of the even integers with the odd integers gives us back the entire set of integers, $\mathbb{Z}/2\mathbb{Z}$ contains only these two cosets.

If we now assign the following labels to our cosets, then we can create the following multiplication table.

$\text{Even } = 2\mathbb{Z} = \{\ldots -4, -2, 0, 2, 4, \ldots\}$

$\text{Odd } = \mathbb{Z} = \{\ldots -5. -3. -1.1.3.5. \ldots\}$

+	**Even**	**Odd**
Even	Even	Odd
Odd	Odd	Even

From this table, it should be clear that our quotient group is isomorphic to the cyclic group of order two which, in turn, is isomorphic to the integers modulo 2, $C_2 \cong \mathbb{Z}_2$. However, there is something else going on here that is extremely important that we haven't really emphasized before. <u>Namely, that once we group the elements from our original group into classes, the elements within a particular class become indistinguishable from one another!</u> For example, at the level of $\mathbb{Z}$, we make distinctions between particular integers such as 5, 10, -20, and 0, but within our quotient group, these distinctions disappear. And with regard to this specific quotient group, $\mathbb{Z}/2\mathbb{Z}$, only two elements remain, even and odd. Everything else is indistinguishable. And this is also why the term "quotient group" is a good name for this structure. In many respects, the quotient group is what results when we "divide" or "cancel" out the differences between the individual elements in our normal subgroup. In this regard, we can even think of something like a chair as a quotient structure. Before we learned as a toddler to call that item a "chair" we probably saw it first as a collection of individual pieces such as legs, seat, etc. However, we quickly learned to divide out the differences between these component parts and see the object as a whole that we label "chair." You can more readily experience this same process today if you start learning a language that uses a different alphabet than the one you are used to. I can guarantee that for quite a while you will have to spell out the letters one by one in order to recognize that you are looking at that language's word for something like "cat." However, if you persevere, you will arrive at the point where you can see their "cat" word as a single whole without having to focus first on the component parts. When you arrive at that point, you have successfully divided out the distinctions between the components so that you can now literally see the forest instead of the individual trees. And this is one of the ways in which we grow intellectually. We recognize that certain pieces go together in some meaningful way, and then our brain creates its own version of a quotient structure by dividing out whatever originally kept the pieces separate from one another.

There is another example of a quotient structure we are going to look at now, and that is the quotient created by the commutator subgroup. Recall that if *a* and *b* are elements in a group *G*, then the commutator of *a* and *b* is $[a,b]=a^{-1}b^{-1}ab$, and the commutator subgroup is the group generated by the product of all such commutators. Recall also that if a group is abelian, then for every *a* and *b* in *G*, we have that $ab=ba$ and this implies that $a^{-1}b^{-1}ab=e$, the identity. Hence, as we mentioned previously, the relative size of the commutator subgroup is a measure of how far our group deviates from being abelian. Additionally, our commutator subgroup is also a normal subgroup, and so if we denote the commutator subgroup of *G* by G', then we can consider the quotient group G/G'. In this quotient group, every commutator of the form $a^{-1}b^{-1}ab$ becomes indistinguishable from the identity since every commutator is an element of G'. In terms of right cosets, this means that $G'(a^{-1}b^{-1}ab)=G'$ and this, in turn, implies that in our quotient group we always have that $e=G'=G'(a^{-1}b^{-1}ab)=G'a^{-1}\cdot G'b^{-1}\cdot G'a\cdot G'b\Rightarrow G'a\cdot G'b=G'b\cdot G'a\cdot$. In other words, our quotient group is abelian, and this is another reason why the commutator subgroup is so important. It's important because we are guaranteed that G/G' is an abelian group. Furthermore, the commutator subgroup of *G*, denoted by G', is also known as the derived subgroup.

And lastly, there is a normal subgroup that always exists that is somewhat complementary to the concept of the commutator (derived) subgroup, and it is called the center of the group. For any group *G*, we can define the center of *G* as the set of all elements that commute with every other element in *G*. At the very least, the identity element is always in the center of a group. And in the case where a group is abelian, its center will be the entire group while the commutator or derived subgroup will consist of just the identity. Hence, the center and the commutator (derived) subgroups are sometimes at opposite ends of the spectrum while in other instances, such as is the case with the Quaternion Group, they are identical.