Lesson 17

MORE ON QUOTIENT GROUPS – ANSWERS

Recall that the rotational and reflexive symmetries of a square give us the dihedral group of degree 4, D_4 = { (), (2,4), (1,2)(3,4), (1,2,3,4), (1,3), (1,3)(2,4), (1,4,3,2), (1,4)(2,3) }, and recall that D_4 contains the subgroups that follow. Decide which one of these subgroups is the commutator subgroup, and also determine the multiplication table for D_4 / D_4' and identify what group you've seen before that this quotient group is isomorphic to.

$$H_1 = D_4 = \{(\,),(2,4),(1,2)(3,4),(1,2,3,4),(1,3),(1,3)(2,4),(1,4,3,2),(1,4)(2,3)\}$$

$$H_2 = \{(\,),(1,3),(2,4),(1,3)(2,4)\}$$

$$H_3 = \{(\,),(1,3)(2,4),(1,2)(3,4),(1,4)(2,3)\}$$

$$H_4 = \{(\,),(1,2,3,4),(1,3)(2,4),(1,4,3,2)\}$$

$$H_5 = \{(\,),(1,3)\}$$

$$H_6 = \{(\,),(1,4)(2,3)\}$$

$$H_7 = \{(\,),(2,4)\}$$

$$H_8 = \{(\,),(1,2)(3,4)\}$$

$$H_9 = \{(\,),(1,3)(2,4)\}$$

$$H_{10} = \{(\,)\}$$

By direct calculation, we can discover that the commutator subgroup is $D_4' = \{(\,),(1,3)(2,4)\}$. Since this subgroup has two elements, the quotient group D_4 / D_4' has four elements (since $\frac{|D_4|}{|D_4'|} = \frac{8}{2} = 4$). We can identify the four right cosets as:

$$D_4' = \left\{ \begin{matrix} (\,) \\ (1,4)(2,3) \end{matrix} \right\}$$

$$D_4'(1,3)=\begin{Bmatrix}(\ )\\(1,4)(2,3)\end{Bmatrix}(1,3)=\begin{Bmatrix}(\ )(1,3)\\(1,4)(2,3)(1,3)\end{Bmatrix}=\begin{Bmatrix}(1,3)\\(1,4,3,2)\end{Bmatrix}$$

$$D_4'(2,4)=\begin{Bmatrix}(\ )\\(1,4)(2,3)\end{Bmatrix}(2,4)=\begin{Bmatrix}(\ )(2,4)\\(1,4)(2,3)(2,4)\end{Bmatrix}=\begin{Bmatrix}(2,4)\\(1,2,3,4)\end{Bmatrix}$$

$$D_4'(1,3)(2,4)=\begin{Bmatrix}(\ )\\(1,4)(2,3)\end{Bmatrix}(1,3)(2,4)=\begin{Bmatrix}(\ )(1,3)(2,4)\\(1,4)(2,3)(1,3)(2,4)\end{Bmatrix}=\begin{Bmatrix}(1,3)(2,4)\\(1,2)(3,4)\end{Bmatrix}$$

For our multiplication table, we have:

*	D_4'	$D_4'(1,3)$	$D_4'(2,4)$	$D_4'(1,3)(2,4)$
D_4'	D_4'	$D_4'(1,3)$	$D_4'(2,4)$	$D_4'(1,3)(2,4)$
$D_4'(1,3)$	$D_4'(1,3)$	D_4'	$D_4'(1,3)(2,4)$	$D_4'(2,4)$
$D_4'(2,4)$	$D_4'(2,4)$	$D_4'(1,3)(2,4)$	D_4'	$D_4'(1,3)$
$D_4'(1,3)(2,4)$	$D_4'(1,3)(2,4)$	$D_4'(2,4)$	$D_4'(1,3)$	D_4'

Additionally, we can see that every element in this quotient group has order 2. Since there are only two groups of order 4, the cyclic group C_4 and the Klein 4-group that is isomorphic to the direct product $C_2 \times C_2$, it follows that since our quotient group has no element of order 4 that the quotient group must be isomorphic to the Klein 4-group, $C_2 \times C_2$.