MATHEMATICAL INDUCTION – PRACTICE

Mathematical induction is a standard proof technique for showing that some proposition *P* about natural numbers holds true for all $n \in \mathbb{N}$ where $\mathbb{N} = \{1, 2, 3, \ldots\}$.

<u>Mathematical Induction:</u> If *P* is a proposition about natural numbers $n \in \mathbb{N}$, then *P* is true for all $n \in \mathbb{N}$ if,

1. *P* is true for $n = 1$, and
2. *P* true for $n \in \mathbb{N} \Rightarrow$ *P* is true for $n+1 \in \mathbb{N}$.

There are several variations we could do of this basic principle. For example, if we began by showing that *P* is true for $n = 0$, then we could possibly prove that *P* is true for all whole numbers. Similarly, if we started our argument by showing that *P* is true for $n = 10$, then a successful induction argument could show that *P* is true for all natural numbers greater than or equal to 10. Another variant form of mathematical induction is shown below.

<u>The Second Principle of Mathematical Induction:</u> If *P* is a proposition about natural numbers $n \in \mathbb{N}$, then *P* is true for all $n \in \mathbb{N}$ if,

3. *P* is true for $n = 1$, and
4. *P* true for all natural numbers less than $n \in \mathbb{N} \Rightarrow$ *P* is true for $n \in \mathbb{N}$.

1. Use mathematical induction to prove that $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$.

2. Use mathematical induction to prove that $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$.

3. Find the flaw in the following inductive argument that all horses are the same color.

   By way of induction, suppose that you have a set containing $n = 1$ *horses. Then clearly all the horses in that set are the same color. Now assume that it is true that in any set of n horses, all the horses have the same color (our induction hypothesis). At this point we want to argue that it is also true that any set of* $n+1$ *horses will also all be the same color. Thus, suppose we are given a set containing* $n+1$ *horses. If we remove one horse, then by our inductive hypothesis the remaining n horses will all be the same color. Now return the horse we originally removed and remove a different horse. Then once again our inductive hypothesis states that the resulting set of n horses all have the same color. From this it follows that the two horses we successively removed have the same color, and therefore, all of the horses in our set of* $n+1$ *horses have the same color. It now follows by mathematical induction that for any set of n horses,* $n \in \mathbb{N}$, *all the horses have the same color.*

4. If A is a set, then the set of all subsets of A, denoted by $P(A)$, is called the power set of A. For example, if $A = \varnothing$, then $P(A) = \{\varnothing\}$. If $A = \{a\}$, then $P(A) = \{\varnothing, \{a\}\}$. And if $A = \{a,b\}$, then $P(A) = \{\varnothing, \{a\}, \{b\}, \{a,b\}\}$. Use mathematical induction to show that if $|A| = n \in \mathbb{N}$, then $|P(A)| = 2^n = 2^{|A|}$.

5. The result from the previous problem not only shows why we call $P(A)$ the power set of A, but also that for any finite set A, $|P(A)| > |A|$. This last result can be extended to infinite sets as well, and this provides a technique for constructing an infinite number of infinite sets of different sizes. In other words, for any infinite set, the cardinality of its power set will be greater than the cardinality of the original set. The smallest infinite set is represented by the set of counting or natural numbers, and we denote the size of this set by $\aleph_0$ (aleph null). Any set of size $\aleph_0$ is called countable or countably infinite. Larger infinite cardinal numbers are denoted by $\aleph_1, \aleph_2, \aleph_3 \ldots$ and so on. Below is a sloppy proof of mine that for any set A, there is no bijection from A to $P(A)$. This shows that the two sets have different cardinalities. However, since we can easily find an injective function from A to $P(A)$, (for instance, if $a \in A$, then pair a with $\{a\} \in P(A)$), it immediately follows that $|P(A)| > |A|$ for any set A. Clean up this proof.

   Theorem: Let A be a set and let $P(A)$ be the set of all subsets of A. Then there is no bijective function from A to $P(A)$, and hence, $|P(A)| > |A|$.

   Sloppy Proof: Let A be a set and let $P(A)$ be the set of all subsets of A. Since the result is obvious when A is empty, assume A is non-empty. Now assume that f is a bijection from A to $P(A)$, and let T be the set of all elements x in A such that x is not an element of $f(x)$. Since f is a bijection, there exists an element t in A such that $f(t) = T$. Now ponder the question is t an element of T? Bummer. Therefore, no bijection exists from A to $P(A)$, and thus, $|P(A)| > |A|$. □

6. Georg Cantor contemplated the set of all sets, but his discovery of the theorem presented in exercise 5 led to a contradiction known as *Cantor's Paradox*. Give an informal discussion of why if U is the set of all sets, then we can reach both the conclusion that $|P(U)| > |U|$ and $|P(U)| \le |U|$.