MATHEMATICAL INDUCTION – ANSWERS

Mathematical induction is a standard proof technique for showing that some proposition *P* about natural numbers holds true for all $n \in \mathbb{N}$ where $\mathbb{N} = \{1, 2, 3, \ldots\}$.

<u>Mathematical Induction:</u> If *P* is a proposition about natural numbers $n \in \mathbb{N}$, then *P* is true for all $n \in \mathbb{N}$ if,

1. *P* is true for $n = 1$, and
2. *P* true for $n \in \mathbb{N} \Rightarrow$ *P* is true for $n+1 \in \mathbb{N}$.

There are several variations we could do of this basic principle. For example, if we began by showing that *P* is true for $n = 0$, then we could possibly prove that *P* is true for all whole numbers. Similarly, if we started our argument by showing that *P* is true for $n = 10$, then a successful induction argument could show that *P* is true for all natural numbers greater than or equal to 10. Another variant form of mathematical induction is shown below.

<u>The Second Principle of Mathematical Induction:</u> If *P* is a proposition about natural numbers $n \in \mathbb{N}$, then *P* is true for all $n \in \mathbb{N}$ if,

3. *P* is true for $n = 1$, and
4. *P* true for all natural numbers less than $n \in \mathbb{N} \Rightarrow$ *P* is true for $n \in \mathbb{N}$.

1. Use mathematical induction to prove that $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$.

   <u>Proof:</u> Let $n = 1$. Then $\frac{1(1+1)}{2} = \frac{2}{2} = 1 = \sum_{k=1}^{1} k$. Hence, the statement is true for $n = 1$.

   Assume now that the statement is true for some natural number *n*, and consider if it is true for $n+1$. Clearly,

   $$\sum_{k=1}^{n+1} k = \left(\sum_{k=1}^{n} k\right) + n + 1 = \frac{n(n+1)}{2} + n + 1 = \frac{n(n+1) + 2(n+1)}{2} = \frac{(n+1)(n+2)}{2} = \frac{(n+1)\left[(n+1)+1\right]}{2}.$$

   Hence, if the formula is true for *n*, then it is also true for $n+1$. Therefore, by mathematical induction, $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$ for all natural numbers *n*. □

2. Use mathematical induction to prove that $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$.

   <u>Proof:</u> Let $n = 1$. Then $\frac{1(1+1)(2+1)}{6} = \frac{6}{6} = 1 = \sum_{k=1}^{1} k^2$. Hence, the statement is true for $n = 1$. Assume now that the statement is true for some natural number *n*, and consider if it is true for $n+1$. Clearly,

$$\sum_{k=1}^{n+1} k^2 = \left(\sum_{k=1}^{n} k^2\right) + (n+1)^2 = \frac{n(n+1)(2n+1)}{6} + \frac{6(n+1)^2}{6} = \frac{(n+1)\left[n(2n+1)+6(n+1)\right]}{6}$$

$$= \frac{(n+1)\left[2n^2+7n+6\right]}{6} = \frac{(n+1)(n+2)(2n+3)}{6} = \frac{(n+1)\left[(n+1)+1\right]\left[2(n+1)+1\right]}{6}.$$

Hence, if the formula is true for *n*, then it is also true for $n+1$. Therefore, by mathematical induction, $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$ for all natural numbers *n*. □

3. Find the flaw in the following inductive argument that all horses are the same color.

   By way of induction, suppose that you have a set containing $n=1$ horses. Then clearly all the horses in that set are the same color. Now assume that it is true that in any set of n horses, all the horses have the same color (our induction hypothesis). At this point we want to argue that it is also true that any set of $n+1$ horses will also all be the same color. Thus, suppose we are given a set containing $n+1$ horses. If we remove one horse, then by our inductive hypothesis the remaining n horses will all be the same color. Now return the horse we originally removed and remove a different horse. Then once again our inductive hypothesis states that the resulting set of n horses all have the same color. From this it follows that the two horses we successively removed have the same color, and therefore, all of the horses in our set of $n+1$ horses have the same color. It now follows by mathematical induction that for any set of n horses, $n \in \mathbb{N}$, all the horses have the same color.

   In the reading of the above argument, one often imagines a case where we might have, for example, 10 horses. We remove one horse, and then our induction hypothesis says that the remaining 9 horses are all the same color. We then replace our first horse, remove another horse, and again our induction hypothesis says that the remaining 9 horses are all the same color. And then finally, we conclude that because of the overlap of the two situations that all 10 horses are the same color. It is, indeed, clear that the induction argument works for the case of $n=10$. However, where the argument breaks down is for $n=2$. When we have 2 horses, then we can remove either one, but the resulting singleton sets this time have no intersection or overlap, and thus, we can't conclude that the two horses have to be of the same color. This is the one break in the chain of the induction argument that at first glance would appear to prove the assertion true for all natural numbers *n*.

4. If *A* is a set, then the set of all subsets of *A*, denoted by $P(A)$, is called the power set of *A*. For example, if $A=\varnothing$, then $P(A)=\{\varnothing\}$. If $A=\{a\}$, then $P(A)=\{\varnothing,\{a\}\}$. And if $A=\{a,b\}$, then $P(A)=\{\varnothing,\{a\},\{b\},\{a,b\}\}$. Use mathematical induction to show that if $|A|=n\in\mathbb{N}$, then $|P(A)|=2^n=2^{|A|}$.

   <u>Proof:</u> Our assertion is actually true for all whole numbers since as we see above, $|\varnothing|=0$ and $P(\varnothing)=\{\varnothing\}\Rightarrow|P(\varnothing)|=1=2^0$. Similarly, if $A=\{a\}$, then $P(A)=\{\varnothing,\{a\}\}$

and $|P(A)| = 2^1 = 2$. Thus, let's assume that for any set with cardinality *n* that the cardinality of its power set is 2^n, and let's suppose that we have a set *A* such that $|A| = n+1 \geq 2$. Then there exists $a \in A$ such that if we remove *a* from *A*, then $|A - \{a\}| = n$, and hence, by our induction hypothesis, $|P(A - \{a\})| = 2^n$. Now consider the structure of $P(A)$. Clearly, every set in $P(A - \{a\})$ also belongs to $P(A)$. Furthermore, we can divide the sets in $P(A)$ into two categories, those that contain *a* and those that don't. A moment's reflection should convince us that *A* should have just as many subsets that contain *a* as don't. For example, we can take any subset that doesn't contain *a* and create one that contains *a* just by adding *a* to it. Similarly, we could start with any subset containing *a* and delete *a* from this subset to obtain one that is lacking *a*. Thus, in $P(A)$ there is a one-to-one correspondence between those subsets that contain *a* and those that don't, and from this it follows that $|P(A)| = 2 \cdot |P(A) - \{a\}| = 2 \cdot 2^n = 2^{n+1} = 2^{|A|}$. Therefore, the assertion is true for all natural numbers, and, indeed, all whole numbers. □

5. The result from the previous problem not only shows why we call $P(A)$ the power set of *A*, but also that for any finite set *A*, $|P(A)| > |A|$. This last result can be extended to infinite sets as well, and this provides a technique for constructing an infinite number of infinite sets of different sizes. In other words, for any infinite set, the cardinality of its power set will be greater than the cardinality of the original set. The smallest infinite set is represented by the set of counting or natural numbers, and we denote the size of this set by $\aleph_0$ (aleph null). Any set of size $\aleph_0$ is called countable or countably infinite. Larger infinite cardinal numbers are denoted by $\aleph_1, \aleph_2, \aleph_3 \ldots$ and so on. Below is a sloppy proof of mine that for any set *A*, there is no bijection from *A* to $P(A)$. This shows that the two sets have different cardinalities. However, since we can easily find an injective function from *A* to $P(A)$, (for instance, if $a \in A$, then pair *a* with $\{a\} \in P(A)$), it immediately follows that $|P(A)| > |A|$ for any set *A*. Clean up this proof.

<u>Theorem:</u> Let *A* be a set and let $P(A)$ be the set of all subsets of *A*. Then there is no bijective function from *A* to $P(A)$, and hence, $|P(A)| > |A|$.

<u>Sloppy Proof:</u> Let *A* be a set and let $P(A)$ be the set of all subsets of *A*. Since the result is obvious when *A* is empty, assume *A* is non-empty. Now assume that *f* is a bijection from *A* to $P(A)$, and let *T* be the set of all elements *x* in *A* such that *x* is not an element of *f(x)*. Since *f* is a bijection, there exists an element *t* in *A* such that $f(t) = T$. Now ponder the question is *t* an element of *T?* Bummer. Therefore, no bijection exists from *A* to $P(A)$, and thus, $|P(A)| > |A|$. □

Proof: Let A be a set and let $P(A)$ be the set of all subsets of A. Since the result is obvious when A is empty, assume A is non-empty. Now assume that f is a bijection from A to $P(A)$, and let T be the set of all elements x in A such that x is not an element of $f(x)$. Since f is a bijection, there exists an element t in A such that $f(t) = T$. Now ponder the question is t an element of T? If $t \in T$, then since T is defined as the set of all elements x in A such that x is not an element of $f(x)$, it follows that $t \notin T$. But on the other hand, if $t \notin T$, then it follows from the definition of T that $t \in T$. Either way we go, we arrive at a contradiction, and the source of these contradictions is the assumption that we have a bijective function f from A to $P(A)$. Hence, no such bijection can exist, and so $|P(A)| \neq |A|$. On the other hand, the function $f : A \to P(A)$ defined by $f(a) = \{a\}$ is clearly an injection, and therefore, $|P(A)| > |A|$. □

6. Georg Cantor contemplated the set of all sets, but his discovery of the theorem presented in exercise 5 led to a contradiction known as *Cantor's Paradox*. Give an informal discussion of why if U is the set of all sets, then we can reach both the conclusion that $|P(U)| > |U|$ and $|P(U)| \leq |U|$.

   By the theorem proved in exercise 5, we know that $|P(U)| > |U|$. But on the other hand, if U is the set of all sets, then it must contain every element in $P(U)$ and that means that there is an obvious injection from $P(U) \to U$. Hence, it must also be true that $|P(U)| \leq |U|$.

   Naively, we think of a set as a collection of objects, but paradoxes such as Russell's Paradox and Cantor's Paradox show us that we have to be more careful about what we are allowed to call a set. This has resulted in axiomatic versions of set theory with the most popular one being known as *Zermelo-Frankel-Axiom of Choice* or *ZFC*. A fairly non-technical version of the axioms for set theory is given below:

   a. Two sets are identical if they have the same elements.
   b. The empty set exists.
   c. If A and B are sets, then $\{A, B\}$ is a set.
   d. The union of a set of sets is a set.
   e. Infinite sets exist.
   f. A property that can be formalized in the language of the theory can be used to define a set.
   g. The power set of a set is a set.
   h. If a set has an element in it, then we can "choose" that element. (Axiom of Choice)
   i. If A is a set, then $A \notin A$.

The above axioms help us avoid the kinds of paradoxes and conundrums that appeared early on in set theory by limiting what we can now call a set, and any collection that is not a *set* is now called a *class*. However, don't think that philosophical problems don't remain. After all, in a sense all we have done is to simply say that it's forbidden to talk about something like “the set of all sets.” Aside from the paradoxes that arise, we still haven’t adequately explained why we can’t talk about the set of everything.

“Do I contradict myself? Very well, then I contradict myself, I am large, I contain multitudes.”
-*Walt Whitman, Leaves of Grass*