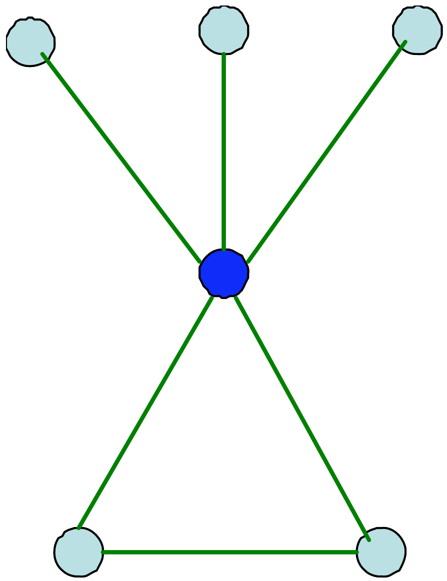
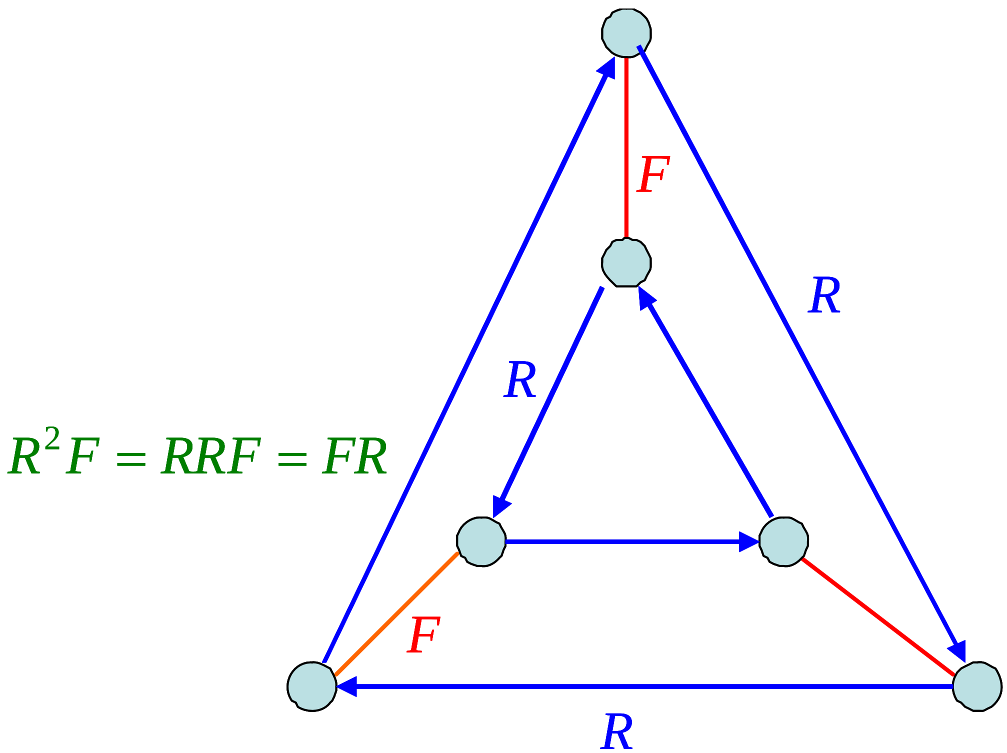
VISUAL GROUP THEORY



Many people find group theory pretty difficult because the logic behind it seems algebraic to the extreme and geometric not at all. However, there are a few ways to visualize groups that we'll now discuss. The first way is by constructing what we call a Cayley graph. To do this, we usually start with a minimal set of generators for the group, and then we create the different elements by multiplying them together. We connect our elements by lines that are colored differently to represent the different generators, and when we are done, the result is a wonderful, symmetrical diagram.

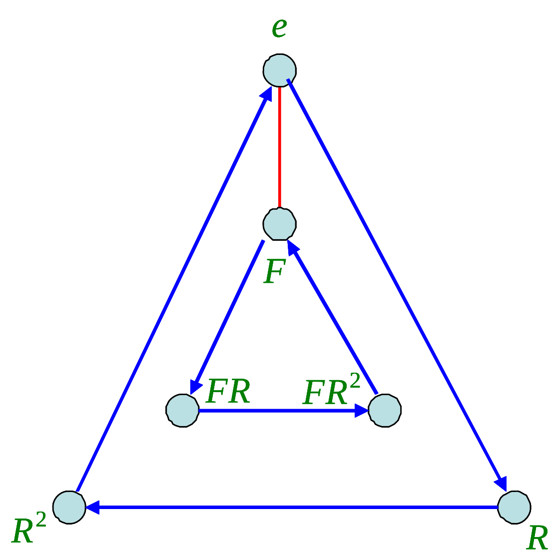
Below is a Cayley diagram for D_3 where I've used blue for the rotation and red for the flip. I've also labeled a few lines so that you can easily see the relation that we discovered earlier that *FR*=*RRF*.



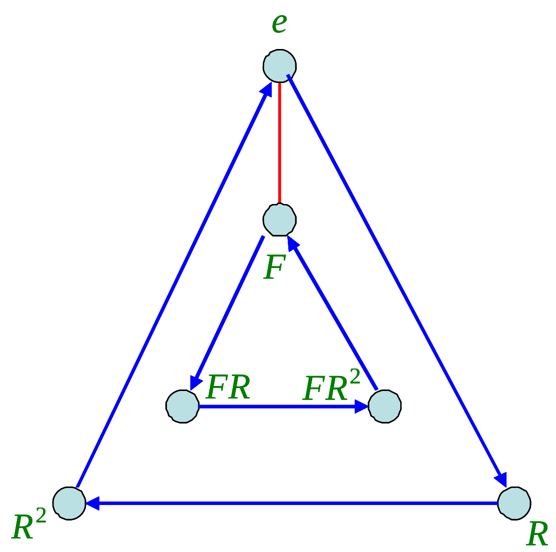
Now let me give a few more details on how to construct such a diagram. First, let's start with the multiplication table for D_3.

	e	R	R^2	F	FR	FR^2
e	e	R	R^2	F	FR	FR^2
R	R	R^2	e	FR^2	F	FR
R^2	R^2	e	R	FR	FR^2	F
F	F	FR	FR^2	e	R	R^2
FR	FR	FR^2	F	R^2	e	R
FR^2	FR^2	F	FR	R	R^2	e

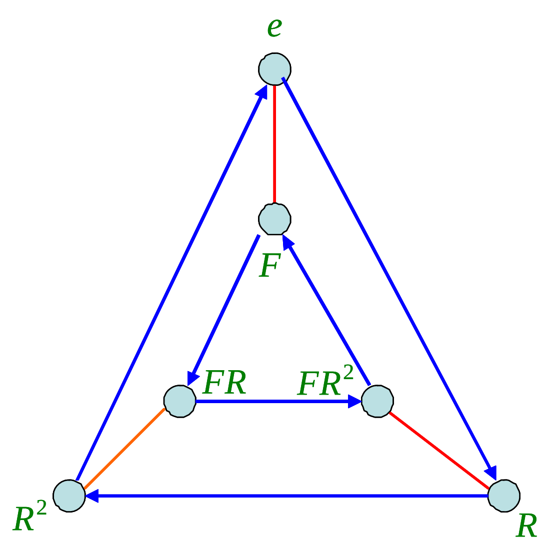
We'll begin our diagram with one dot for *e*, and then emerging from this dot we'll draw a blue arrow for *R* that we'll let terminate at a dot with the same name. Next, we construct another blue arrow emanating from the dot labeled *R*, and we let this arrow terminate at a dot that we'll label R^2. If we repeat this process one more time, then that will take us back to *e* since $R^3 = e$.



We now begin again at e with a red arrow for F that, again, we'll terminate at a dot with the same name. Also, since F has order 2, it's customary not to put an arrow head on this one. At this point, there are just two elements left in our multiplication table that are not yet in our diagram, FR and FR^2. Hence, construct a blue line emerging from the dot labeled F, and let it terminate at a dot that we'll label FR. And as before, add a blue arrow to FR, and let it terminate at FR^2. And then as you would expect, a blue arrow emanating from FR^2 will take us back to F since $FR^3 = F$. At this point, your connected dots and arrows should look like the diagram below.

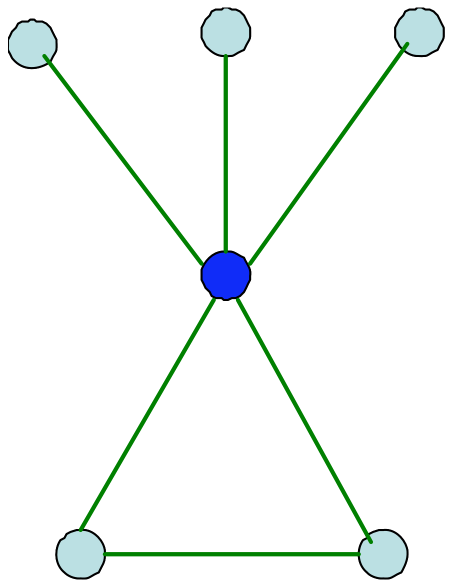


And now there is just one thing missing. Since our group is generated by the elements R and F, every dot in our Cayley graph needs both a red arrow and a blue arrow emanating from it to show us where the multiplication takes us. However, since we know that all the elements in our group are now represented by dots in the diagram, all we need to do is to look at our multiplication table above to see where the red arrows will take us if we follow each of FR and FR^2 by F. And below is our completed Cayley graph!

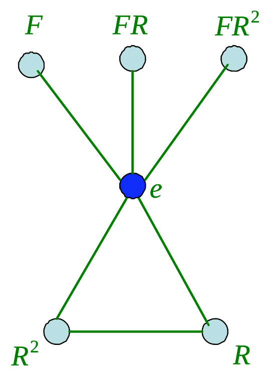


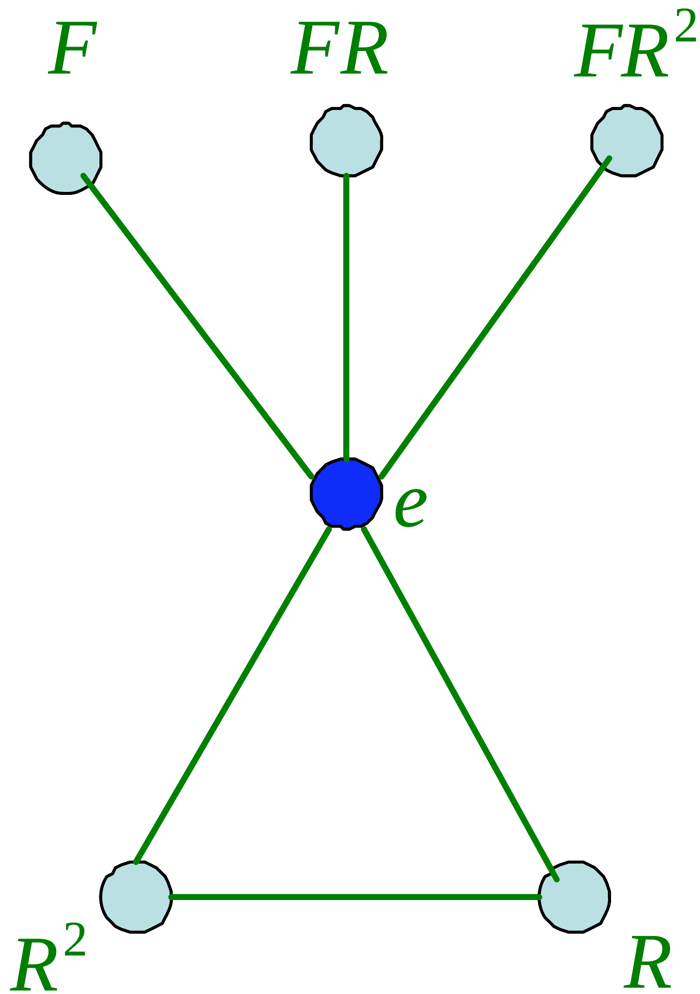
The second diagram I want to show you is called a cycle graph, and I really like it better. Basically, for every element in your group, you make a sketch that shows the cycle it creates. I like this because I think one of the most important things to realize about group theory is that all the complex structures we might create are created by simply combing cycles in various ways. Also, if we have something like a cycle of length 4 in our graph, then that's clearly going to contain a cycle of length 2. In this case, we don't make a separate diagram for the 2-cycle. We just recognize that it's already part of the 4-cycle.

Below now is a cycle graph for D_3. Recall that it has one cyclic subgroup of order 3 and three separate cyclic subgroups of order 2. Also, the blue dot in the middle represents the identity element in the group. Think of it as your starting point!



As with the Cayley graphs, let's give a little more detail on how to do the construction using the multiplication table for D_3 that we presented earlier. Again we start with one dot for *e*. We now go to the element *R* in our table and note that it has order 3. This is going to result in a cycle of length 3 in our cycle graph. Additionally, since R^2 is part of this cycle, we can now skip in our table to *F*. Since the order of *F* is 2, we draw a visual representation of a 2-cycle that starts at *e*. Similarly, both *FR* and FR^2 have order 2, and so we add two more 2-cycles to our diagram. And now we're done because all the elements of our group are now present in our cycle graph! Below is the final product with the elements labeled.




F
FR
FR2
e
R2
R

Both Cayley graphs and cycle graphs can help us to better understand the structure of a group just so long as its not too large. When we get into larger groups, though, these diagrams get very complicated! Also, one problem with cycle graphs is that once we get to groups of order 16, we can actually have different, non-isomorphic groups that generate the same cycles. Nonetheless, as long as we stay below order 16, each group will have a unique cycle graph. Furthermore, even for larger groups we can use a cycle graph to help show us what groups it’s not isomorphic to.

As a final note, for both cyclic and dihedral groups, the cycle graphs are always pretty simple. For example, the cycle graph for a cyclic group of order *n* can be represented by a regular polygon with *n* sides, and the dihedral group D_n will have one cycle that can be represented by an *n*-sided polygon, and then it will also have *n* additional cycles of order 2. Thus, the cycle graph for D_4 looks like the following (if I do a quick and dirty sketch in PowerPoint). Cycle graphs and Cayley graphs of several groups may be found in the program Group Explorer.