SOME SPECIAL CLASSES OF GROUPS

Cyclic Groups

When we think of cyclic groups, we immediately think of clock arithmetic which is something that most people are familiar with. For example, in clock arithmetic, if you add 3 hours to 9 o'clock, then you get 12 o'clock, and if you then add 1 more hour, then you're back at 1 o'clock. In clock arithmetic, addition causes us to cycle through the same 12 values over and over. The only change that mathematicians like to make, however, is to replace 12 by 0 since that is going to be our identity element. Thus, for mathematicians the set of elements we'll cycle through is $\{0,1,2,3,4,5,6,7,8,9,10,11\}$. Furthermore, we can reach all of these elements simply by starting with 1 and repeatedly adding 1 to itself. This set, however, coupled with this type of addition results in a group and because adding 1 to itself generates a repeating cycle, we call this a *cyclic group*.

Furthermore, suppose we don't have a finite cycle. In other words, suppose we can either add or subtract 1 from itself forever. In this case, we get a group that is identical to the integers, and so we denote this infinite cyclic group by $\mathbb{Z}$, the standard symbol for the integers

On the other hand, the finite cyclic group above with only 12 elements is called the *integers modulo 12*, and it is denoted by $\mathbb{Z}_{12}$. In this group, once we get to the 11 and add 1 to it, we start all over at 0 and repeat the cycle again. Cyclic groups are always abelian groups, and so I’ll use additive rather than multiplicative notation for them.

In particular, let's look at the multiplication tables for $\mathbb{Z}_3$ and $\mathbb{Z}_4$.

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

Multiplication table for $\mathbb{Z}_3$

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

Multiplication table for $\mathbb{Z}_4$

One of the things we can immediately see in each table is symmetry. In other words, in both tables if you draw a line down the diagonal that starts in the upper left corner, then what is on one side of the line is a mirror image of what's on the other side. This symmetry is a direct consequence of the fact that the groups are abelian, and you'll find this kind of symmetry in the multiplication table for any abelian group. This symmetry is an immediate consequence of equalities such as $1+2=2+1$.

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

Not only are cyclic groups very easy to understand, they are also extremely important to the deeper understanding of group theory because, as we’ll see later on, we can consider any group as being constructed from cycles or cyclic groups that interact with one another in various ways.

Permutation Groups

We’ve already been exposed to groups of permutations, and we’ve seen various ways to represent those permutations, and we’ve also learned how to multiply permutations. There are now just two more things we want to understand. First, if you take any set of permutations for a set of objects, and if you begin looking at all the products you can make by multiplying either those permutations or their inverses by one another, then you will generate a permutation group. Thus, permutation groups are easy to build if you are simply given a few permutations to begin with.

The second thing we want to realize is that every group is *isomorphic* to a group of permutations. The word *isomorphic* essentially means "identical shape," and when we say that two groups are isomorphic, that means that they are essentially the same group except for how we might label the elements. In other words, the two groups have to have the same number of elements, and if a and b in one group are called c and d in the other, then the product ab has to correspond to the product cd. When two groups G and H are isomorphic, we also write $G \cong H$.

The statement that every group is isomorphic to a permutation group is called Cayley's Theorem after the mathematician Arthur Cayley (1821 – 1895) who discovered it. To show how this theorem works, let's use the multiplication table for $\mathbb{Z}_4$ as an illustration.

$+$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

Multiplication table for $\mathbb{Z}_4$

What we want to notice is that each row of this table is also a permutation of our original set of elements $\{0,1,2,3\}$. Thus, using this table, we can establish a correspondence between elements of this group and the permutations represented by the rows. The actual correspondence for this group is shown below

$$\begin{array}{lcl} 0 & \rightarrow & (0)(1)(2)(3) \\ 1 & \rightarrow & (0\ \ 1\ \ 2\ \ 3) \\ 2 & \rightarrow & (0\ \ 2)(1\ \ 3) \\ 3 & \rightarrow & (0\ \ 3\ \ 2\ \ 1) \end{array}$$

Furthermore, just as $1+2=3$, so does $\begin{pmatrix}0 & 1 & 2 & 3\end{pmatrix}\begin{pmatrix}0 & 2\end{pmatrix}\begin{pmatrix}1 & 3\end{pmatrix}=\begin{pmatrix}0 & 3 & 2 & 1\end{pmatrix}$.

$$\begin{aligned}
0 &\rightarrow (0)(1)(2)(3) \\
1 &\rightarrow \begin{pmatrix}0 & 1 & 2 & 3\end{pmatrix} \\
2 &\rightarrow \begin{pmatrix}0 & 2\end{pmatrix}\begin{pmatrix}1 & 3\end{pmatrix} \\
3 &\rightarrow \begin{pmatrix}0 & 3 & 2 & 1\end{pmatrix}
\end{aligned}$$

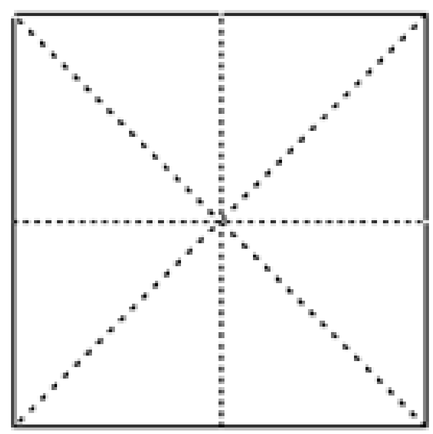
Symmetric Groups

A special type of permutation group is the group of all permutations we can make of *n* objects. We call this the *symmetric group of degree n*, and we denote it by S_n.

Furthermore, as we've previously calculated, $|S_n| = n!$. Thus, the group of permutations we can make of 3 objects is S_3, and the order or number of elements in S_3 is $|S_3| = 3! = 3 \cdot 2 \cdot 1 = 6$. Also, whenever we have a set of objects and a group of permutations for those objects, we like to say that the group *acts* on those objects. In more complete books on group theory, you'll find a much more formal definition of what a *group action* is, but what I've said above is the essence of a group acting on a set of objects.

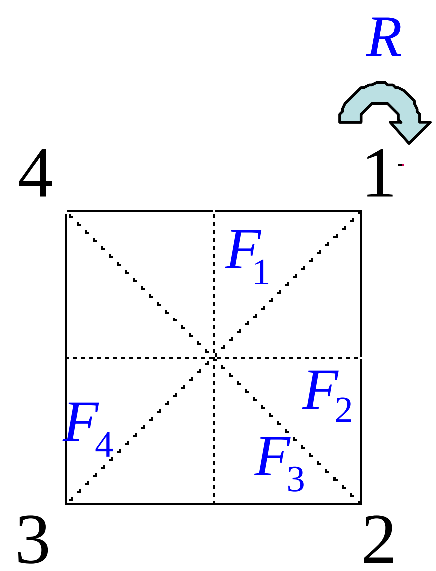
Dihedral Groups

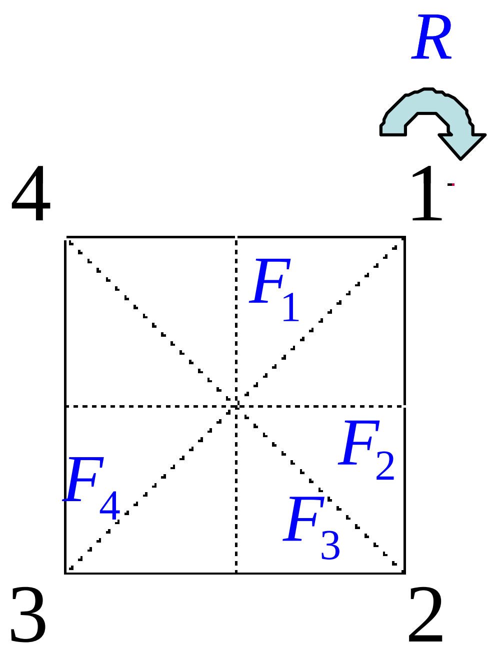
The last special class of groups we want to look at are the *dihedral groups*, and these groups are going to be good examples because they involve both geometric symmetry and permutations. We'll illustrate what a dihedral group is by looking at D_4, the dihedral group of order 8. This group is based on the isometries of a square such as the one below



Recall that an isometry is a movement that leaves the object looking the same as what you started with and that also preserves distances between points. In the case of the square, we can either rotate it clockwise about the center through angles of $90°$, or we can flip it about one of the axes of symmetry that are denoted above by the dotted lines. If we denote doing nothing to our square by *e*, a clockwise $90°$ rotation by *R*, and flips about the four axes of symmetry by F_1, F_2, F_3, and F_4, then the eight elements of our group are $\{e, R, R^2, R^3, F_1, F_2, F_3, F_4\}$. Also, the word dihedral means "two sides," and we call these dihedral groups because, in addition to the rotations, we flip things over from one side to the other.

It's clear that the permissible elements of this group are based upon the symmetries of the square. Furthermore, if we label our vertices 1, 2, 3, and 4, then we can keep track of how each element of the group acts upon the square, and this also shows us that each element of the group also corresponds to a particular permutation of these numbers. In particular, by examining the diagram below, you'll be able to see how we come up with each of our permutations.





$$
\begin{aligned}
e &= (1)(2)(3)(4) \\
R &= \begin{pmatrix}1 & 2 & 3 & 4\end{pmatrix} \\
R^2 &= \begin{pmatrix}1 & 3\end{pmatrix}\begin{pmatrix}2 & 4\end{pmatrix} \\
R^3 &= \begin{pmatrix}1 & 4 & 3 & 2\end{pmatrix} \\
F_1 &= \begin{pmatrix}1 & 4\end{pmatrix}\begin{pmatrix}2 & 3\end{pmatrix} \\
F_2 &= \begin{pmatrix}1 & 2\end{pmatrix}\begin{pmatrix}3 & 4\end{pmatrix} \\
F_3 &= \begin{pmatrix}1 & 3\end{pmatrix} \\
F_4 &= \begin{pmatrix}2 & 4\end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
e &= (1)(2)(3)(4) \\
R &= \begin{pmatrix}1 & 2 & 3 & 4\end{pmatrix} \\
R^2 &= \begin{pmatrix}1 & 3\end{pmatrix}\begin{pmatrix}2 & 4\end{pmatrix} \\
R^3 &= \begin{pmatrix}1 & 4 & 3 & 2\end{pmatrix} \\
F_1 &= \begin{pmatrix}1 & 4\end{pmatrix}\begin{pmatrix}2 & 3\end{pmatrix} \\
F_2 &= \begin{pmatrix}1 & 2\end{pmatrix}\begin{pmatrix}3 & 4\end{pmatrix} \\
F_3 &= \begin{pmatrix}1 & 3\end{pmatrix} \\
F_4 &= \begin{pmatrix}2 & 4\end{pmatrix}
\end{aligned}
$$

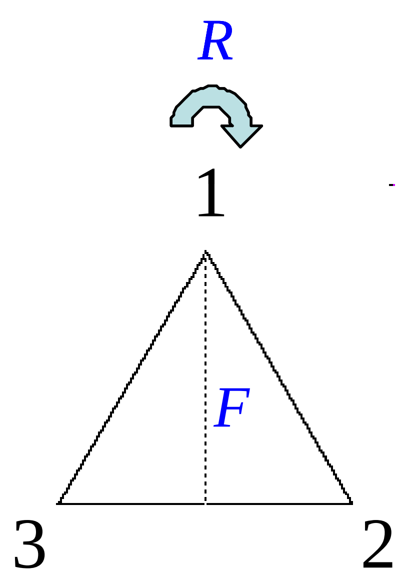
Now notice that:

$$F_1R = \begin{pmatrix}1 & 4\end{pmatrix}\begin{pmatrix}2 & 3\end{pmatrix}\begin{pmatrix}1 & 2 & 3 & 4\end{pmatrix} = \begin{pmatrix}2 & 4\end{pmatrix} = F_4$$

$$F_1R^2 = \begin{pmatrix}1 & 4\end{pmatrix}\begin{pmatrix}2 & 3\end{pmatrix}\begin{pmatrix}1 & 3\end{pmatrix}\begin{pmatrix}2 & 4\end{pmatrix} = \begin{pmatrix}1 & 2\end{pmatrix}\begin{pmatrix}3 & 4\end{pmatrix} = F_2$$

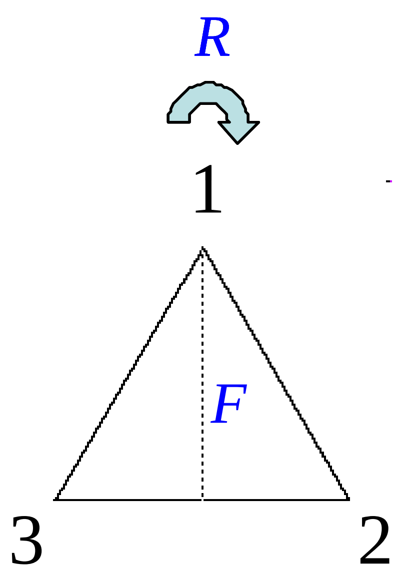
$$F_1R^3 = \begin{pmatrix}1 & 4\end{pmatrix}\begin{pmatrix}2 & 3\end{pmatrix}\begin{pmatrix}1 & 4 & 3 & 2\end{pmatrix} = \begin{pmatrix}1 & 3\end{pmatrix} = F_3$$

What this last set of computations means is that every element in our dihedral group can be generated by the successive rotations combined with a single flip, and this is true for every dihedral group. Thus, when creating a multiplication table for a dihedral group, we usually express everything in terms of products of rotations and a single flip. As an illustration, we'll examine the multiplication table for D_3, the symmetries of a regular triangle.



$$
\begin{aligned}
e &= (1)(2)(3) \\
R &= \begin{pmatrix}1 & 2 & 3\end{pmatrix} \\
R^2 &= \begin{pmatrix}1 & 3 & 2\end{pmatrix} \\
F &= \begin{pmatrix}2 & 3\end{pmatrix} \\
FR &= \begin{pmatrix}1 & 2\end{pmatrix} \\
FR^2 &= \begin{pmatrix}1 & 3\end{pmatrix}
\end{aligned}
$$

We can write the elements of our group as $\{e, R, R^2, F, FR, FR^2\}$. These group elements correspond to the following permutations of the labeled vertices of the triangle:



$$e = (1)(2)(3)$$
$$R = (1\ 2\ 3)$$
$$R^2 = (1\ 3\ 2)$$
$$F = (2\ 3)$$
$$FR = (1\ 2)$$
$$FR^2 = (1\ 3)$$

At this point, we might want to look back at the multiplication table that was presented at the beginning of our slideshow on “What is a Group?”

	$(1)(2)(3)$	$(1\ 2)$	$(1\ 3)$	$(2\ 3)$	$(1\ 2\ 3)$	$(1\ 3\ 2)$
$(1)(2)(3)$	$(1)(2)(3)$	$(1\ 2)$	$(1\ 3)$	$(2\ 3)$	$(1\ 2\ 3)$	$(1\ 3\ 2)$
$(1\ 2)$	$(1\ 2)$	$(1)(2)(3)$	$(1\ 2\ 3)$	$(1\ 3\ 2)$	$(1\ 3)$	$(2\ 3)$
$(1\ 3)$	$(1\ 3)$	$(1\ 3\ 2)$	$(1)(2)(3)$	$(1\ 2\ 3)$	$(2\ 3)$	$(1\ 2)$
$(2\ 3)$	$(2\ 3)$	$(1\ 2\ 3)$	$(1\ 3\ 2)$	$(1)(2)(3)$	$(1\ 2)$	$(1\ 3)$
$(1\ 2\ 3)$	$(1\ 2\ 3)$	$(2\ 3)$	$(1\ 2)$	$(1\ 3)$	$(1\ 3\ 2)$	$(1)(2)(3)$
$(1\ 3\ 2)$	$(1\ 3\ 2)$	$(1\ 3)$	$(2\ 3)$	$(1\ 2)$	$(1)(2)(3)$	$(1\ 2\ 3)$

	$(1)(2)(3)$	$(1\ 2)$	$(1\ 3)$	$(2\ 3)$	$(1\ 2\ 3)$	$(1\ 3\ 2)$
$(1)(2)(3)$	$(1)(2)(3)$	$(1\ 2)$	$(1\ 3)$	$(2\ 3)$	$(1\ 2\ 3)$	$(1\ 3\ 2)$
$(1\ 2)$	$(1\ 2)$	$(1)(2)(3)$	$(1\ 2\ 3)$	$(1\ 3\ 2)$	$(1\ 3)$	$(2\ 3)$
$(1\ 3)$	$(1\ 3)$	$(1\ 3\ 2)$	$(1)(2)(3)$	$(1\ 2\ 3)$	$(2\ 3)$	$(1\ 2)$
$(2\ 3)$	$(2\ 3)$	$(1\ 2\ 3)$	$(1\ 3\ 2)$	$(1)(2)(3)$	$(1\ 2)$	$(1\ 3)$
$(1\ 2\ 3)$	$(1\ 2\ 3)$	$(2\ 3)$	$(1\ 2)$	$(1\ 3)$	$(1\ 3\ 2)$	$(1)(2)(3)$
$(1\ 3\ 2)$	$(1\ 3\ 2)$	$(1\ 3)$	$(2\ 3)$	$(1\ 2)$	$(1)(2)(3)$	$(1\ 2\ 3)$

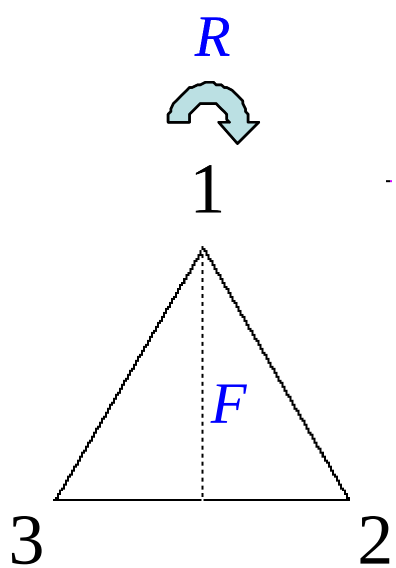
The table from our previous chapter is the multiplication table for the group of all permutations we can make of three objects, the group that we have now identified as S_3, the symmetric group of degree 3. However, this group contains the exact same permutations that are associated with D_3, and that means that these two groups are isomorphic. In other words, they are really the same group, just with different labels and arising from different contexts.

	$(1)(2)(3)$	$(1\ 2)$	$(1\ 3)$	$(2\ 3)$	$(1\ 2\ 3)$	$(1\ 3\ 2)$
$(1)(2)(3)$	$(1)(2)(3)$	$(1\ 2)$	$(1\ 3)$	$(2\ 3)$	$(1\ 2\ 3)$	$(1\ 3\ 2)$
$(1\ 2)$	$(1\ 2)$	$(1)(2)(3)$	$(1\ 2\ 3)$	$(1\ 3\ 2)$	$(1\ 3)$	$(2\ 3)$
$(1\ 3)$	$(1\ 3)$	$(1\ 3\ 2)$	$(1)(2)(3)$	$(1\ 2\ 3)$	$(2\ 3)$	$(1\ 2)$
$(2\ 3)$	$(2\ 3)$	$(1\ 2\ 3)$	$(1\ 3\ 2)$	$(1)(2)(3)$	$(1\ 2)$	$(1\ 3)$
$(1\ 2\ 3)$	$(1\ 2\ 3)$	$(2\ 3)$	$(1\ 2)$	$(1\ 3)$	$(1\ 3\ 2)$	$(1)(2)(3)$
$(1\ 3\ 2)$	$(1\ 3\ 2)$	$(1\ 3)$	$(2\ 3)$	$(1\ 2)$	$(1)(2)(3)$	$(1\ 2\ 3)$

$$e = (1)(2)(3)$$
$$R = (1\ 2\ 3)$$
$$R^2 = (1\ 3\ 2)$$
$$F = (2\ 3)$$
$$FR = (1\ 2)$$
$$FR^2 = (1\ 3)$$

	e	R	R^2	F	FR	FR^2
e	e	R	R^2	F	FR	FR^2
R	R	R^2	e	FR^2	F	FR
R^2	R^2	e	R	FR	FR^2	F
F	F	FR	FR^2	e	R	R^2
FR	FR	FR^2	F	R^2	e	R
FR^2	FR^2	F	FR	R	R^2	e

As a final comment on dihedral groups, we denote the dihedral group associated with a regular polygon with n vertices by D_n, and the order of D_n is $|D_n| = 2n$.



$$
\begin{aligned}
e &= (1)(2)(3) \\
R &= (1\ \ 2\ \ 3) \\
R^2 &= (1\ \ 3\ \ 2) \\
F &= (2\ \ 3) \\
FR &= (1\ \ 2) \\
FR^2 &= (1\ \ 3)
\end{aligned}
$$

$$\{e, R, R^2, F, FR, FR^2\}$$