RUBIK'S CUBE SUBGROUPS

The tools we have developed can now give us more information about the kinds of subgroups that exist within all the permutations that may be reached on Rubik's cube. For example, we previously showed that the total number of attainable permutations is $43,252,003,274,489,856,000$. This rather large number factors into $43,252,003,274,489,856,000 = 2^{27} \cdot 3^{14} \cdot 5^{3} \cdot 7^{2} \cdot 11$. This immediately tells us several things. For instance, there will be Sylow p-subgroups with orders of 2^{27}, 3^{14}, 5^{3}, 7^{2}, and 11. In fact, here is a generator for one of the Sylow 11-subgroups. Interesting, isn't it!

$$U^{-1}FBU^{-1}F^{-1}DBUDB^{-1}U^{-1}RRD^{-1}LLU^{-1}LLD^{-1}LLU^{-1}R$$

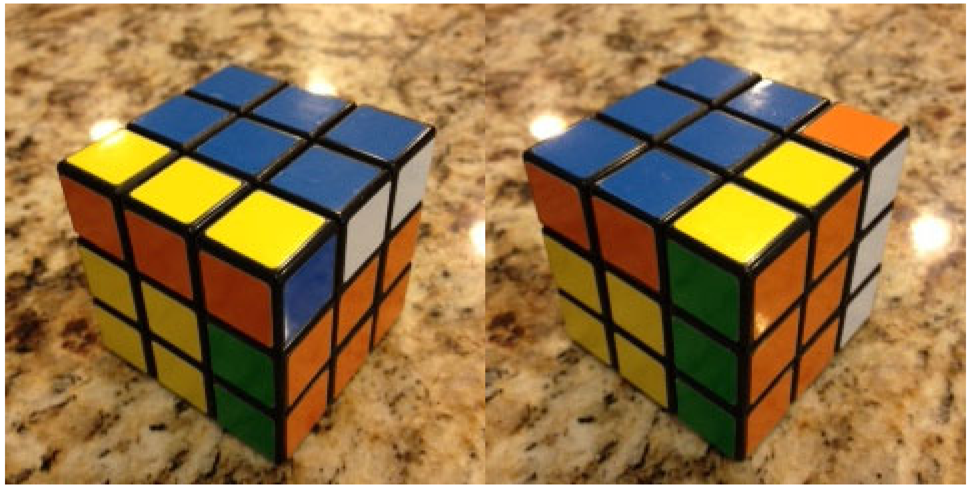
There will be additional subgroups of orders 2, 3, 5, and 7 raised to all the various powers between 1 and the power of the corresponding Sylow p-subgroup. And then there will undoubtedly be a whole lot of other subgroups whose orders are not simply a prime raised to a power. However, we know immediately that there is no subgroup of order 13. And how do we know this? Simple! It's because 13 doesn't divide the order of the group

$$43{,}252{,}003{,}274{,}489{,}856{,}000 = 2^{27} \cdot 3^{14} \cdot 5^{3} \cdot 7^{2} \cdot 11$$

The cube group itself is generated by the operations of *R*, *L*, *U*, *D*, *F*, and *B* being applied to the cube, and each individual operation generates a cyclic group of order 4. For example, $\langle R \rangle = \{e, R, R^2, R^3\}$ is the subgroup that we generate by rotating the right face of the cube a quarter turn each time, and this subgroup is isomorphic to $\mathbb{Z}_4$. Likewise, $\langle L \rangle$, $\langle U \rangle$, $\langle D \rangle$, $\langle F \rangle$, and $\langle B \rangle$ are all isomorphic to $\mathbb{Z}_4$.

Rotating the right face a quarter-turn four times

If we look at the subgroup that is generated by both *R* and *L*, i.e. by twisting the right and left faces separately, then since these operations commute with one another we get $\langle R, L \rangle \cong \mathbb{Z}_4 \oplus \mathbb{Z}_4$. In other words, we can think of the group that is generated by *R* and *L* as simply consisting of ordered pairs where the elements of *R* might occupy the first coordinate, and then the elements of *L* can occupy the second coordinate. It's a nice, simple, abelian group. However, if we perform the move *RU* on the cube, then we are working with cycles that overlap, and the result is far from abelian, or, to put it another way, $RU \neq UR$. Furthermore, if we keep repeating this move *RU*, then we eventually generate a cyclic group of order 105!



$$RU \neq UR$$

Things work out a little differently, though, if we repeatedly do the operation R^2U^2. For one thing, if we look at the cycle structure of just the permuted cublets and ignore any rotations or flips that might occur along the way, then we can describe it by

$$(UB \quad UF)(BR \quad FR)(UL \quad UR \quad DR)(UBL \quad UFR \quad DBR)(ULF \quad URB \quad DRF).$$

R^2U^2

Notice that we have two cycles of length 2 and three cycles of length 3. This means that if we do this operation twice, $\left(R^2U^2\right)^2$, then we will undo the 2-cycles and just be left with some 3-cycles. In fact, the resulting permutation is

$$(DR \quad UR \quad UL)(DBR \quad UFR \quad UBL)(DRF \quad URB \quad ULF)$$

To see this algebraically, let's just raise our first permutation to the second power. If we do, then we'll get

$$\left[(UB \quad UF)(BR \quad FR)(UL \quad UR \quad DR)(UBL \quad UFR \quad DBR)(ULF \quad URB \quad DRF)\right]^2$$
$$=(UB \quad UF)^2(BR \quad FR)^2(UL \quad UR \quad DR)^2(UBL \quad UFR \quad DBR)^2(ULF \quad URB \quad DRF)^2$$
$$=(UL \quad UR \quad DR)^2(UBL \quad UFR \quad DBR)^2(ULF \quad URB \quad DRF)^2$$
$$=(DR \quad UR \quad UL)(DBR \quad UFR \quad UBL)(DRF \quad URB \quad ULF).$$

$(R^2U^2)^2$

If we cube R^2U^2, however, then we'll get rid of the 3-cycles and we'll be left with only a couple of 2-cycles. Algebraically, the result is

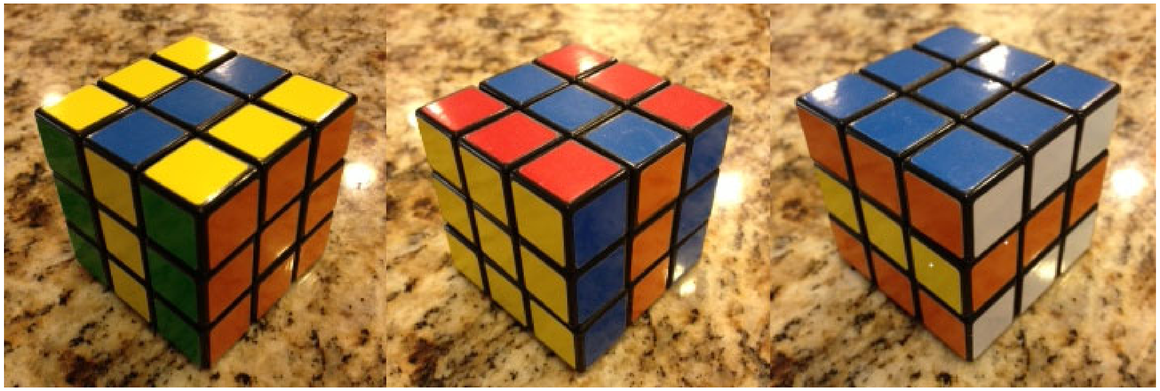
$$
\begin{aligned}
\left(R^2U^2\right)^3 &= \left[\left(UB \quad UF\right)\left(BR \quad FR\right)\left(UL \quad UR \quad DR\right)\left(UBL \quad UFR \quad DBR\right)\left(ULF \quad URB \quad DRF\right)\right]^3 \\
&= \left(UB \quad UF\right)^3\left(BR \quad FR\right)^3\left(UL \quad UR \quad DR\right)^3\left(UBL \quad UFR \quad DBR\right)^3\left(ULF \quad URB \quad DRF\right)^3 \\
&= \left(UB \quad UF\right)\left(BR \quad FR\right).
\end{aligned}
$$

$(R^2U^2)^3$

This final result looks particularly useful because essentially we are just swapping two back cubelets for two front cublelets, and if you try this move, then you'll you get a very nice and elegant pattern. And lastly, since R^2U^2 results in a combination of 2-cycles and 3-cycles, it follows that if we perform this operation six times, then all the cubelets will be restored to their original positions. When we try it, that is indeed what happens, and fortunately the orientations of the cubelets are also restored. Thus, the order of the cyclic group generated by R^2U^2 is six. In symbols, we write $\left|\left\langle R^2U^2\right\rangle\right| = 6$. One of the very important lessons from this example, however, is that looking at the cycle structure of a permutation can help us determine not only the order of the corresponding cyclic group, but also what powers of this permutation might result in moving only a minimum number of cubelets in our Rubik's cube.

And finally, if we look at not only the cyclic group generated by R^2U^2, but also the group generated by R^2 and U^2 (denoted by $\langle R^2, U^2 \rangle$) acting either together or independently, then it turns out that this group is isomorphic to D_6, the symmetries of a regular hexagon. This is also an example of what on the cube we call a *two-squares group*.

Another subgroup of the cube group that is both simple and interesting is called the *slice group*. This subgroup is generated by rotating only the center slices, and as such, it will leave the corners of the cube untouched. Consequently, this group can be used to create some pretty patterns. Also, since it is not always easy to rotate a middle slice, we can accomplish the same effect by performing RL^{-1}, FB^{-1}, and UD^{-1}. Thus, the slice group is generated by these elements, $\langle RL^{-1}, FB^{-1}, UD^{-1} \rangle$.



$RL^{-1}, FB^{-1}, UD^{-1}$

Also interesting and mathematically simpler is the *slice-squared group*,

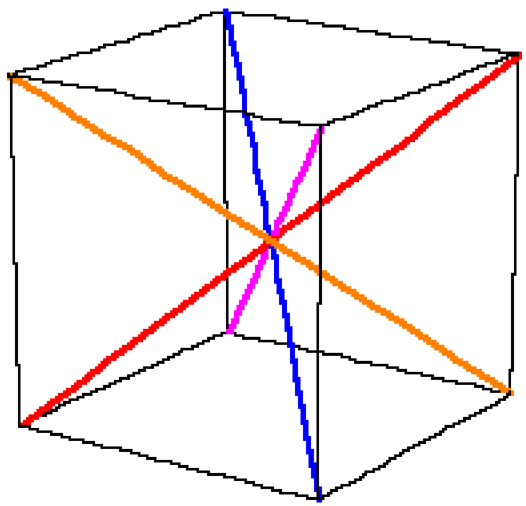
$$\left\langle (RL^{-1})^2, (FB^{-1})^2, (UD^{-1})^2 \right\rangle = \left\langle R^2L^{-2}, F^2B^{-2}, U^2D^{-2} \right\rangle = \left\langle R^2L^2, F^2B^2, U^2D^2 \right\rangle$$

$$R^2L^2, F^2B^2, U^2D^2$$

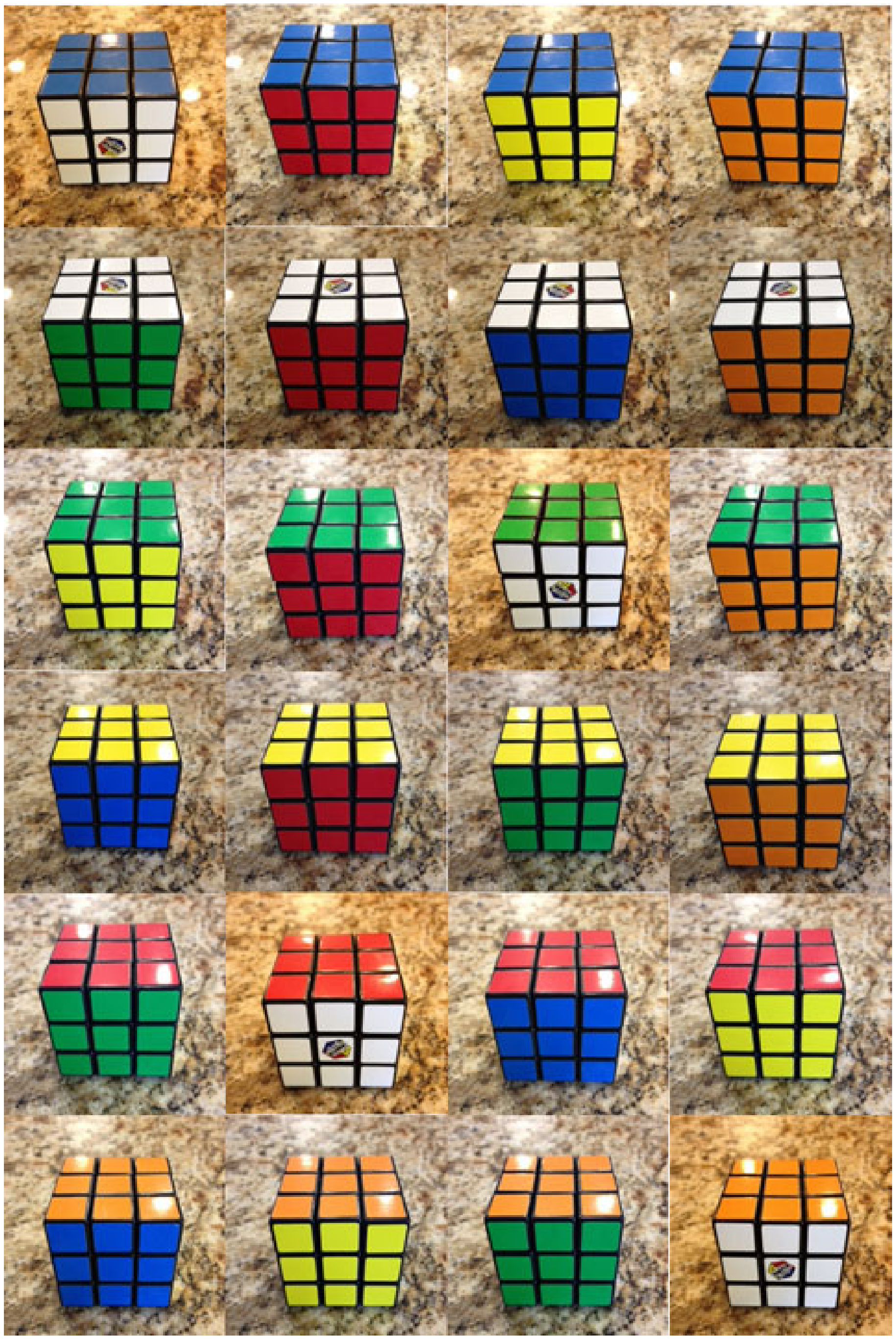
This group consists of eight elements, and it's abelian. And that means, by the Fundamental Theorem of Finite Abelian Groups, there are only three possibilities for the structure of the slice-squared group. It has to be isomorphic to either $\mathbb{Z}_8$, $\mathbb{Z}_4 \times \mathbb{Z}_2$, or $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. See if you can figure out what the correct answer is!

And finally, I want to talk about just one more group that I can associate with Rubik's cube. This is going to be the group generated by rotating the whole cube clockwise with respect to either the up face, the right face, or the front face. I'll represent quarter turns in each of these directions by **U**, **R**, and **F**. Since these moves create a permutation of the six faces of the cube, the group generated has to be some subgroup of S_6 which has order $6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$. However, we won't get S_6 in its entirety. In fact, I claim that our subgroup will only have order 24. To see this, notice that we have six choices we could make regarding which colored face to have at the top of our cube. However, once we have picked a top color, then we have four choices for the front color, and once we have made these two choices, then we're done. Those two choices will establish a particular arrangement for the six faces of the cube. Thus, the total number of arrangements we can have is $6 \cdot 4 = 24$.

Another way to look at this is to construct the four possible diagonals that can go from a bottom corner of the cube to a top corner of the cube, and let's suppose we give each diagonal a different color, such as red, blue, orange, or magenta.



Then every turn of the cube by **U**, **R**, or **F** will produce some permutation of these four diagonals, and the total number of permutations possible is $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$. Furthermore, notice that $\mathbf{U^{-1}RU}$ is equivalent to **F**. Thus, we could generate this group using only **U** and **R**, but it's conceptually easier to think of it as being generated by **U**, **R**, and **F**. Also, no one has named this particular group, and so I recommend calling it **Benton's Group.** (Hey! I need the fame!) And lastly, since $|S_4| = 24$, it turns out that **Benton's Group** is isomorphic to S_4. Sweet!



Benton's Group $\cong S_4$

$|S_4| = 24$

(Make me famous!)