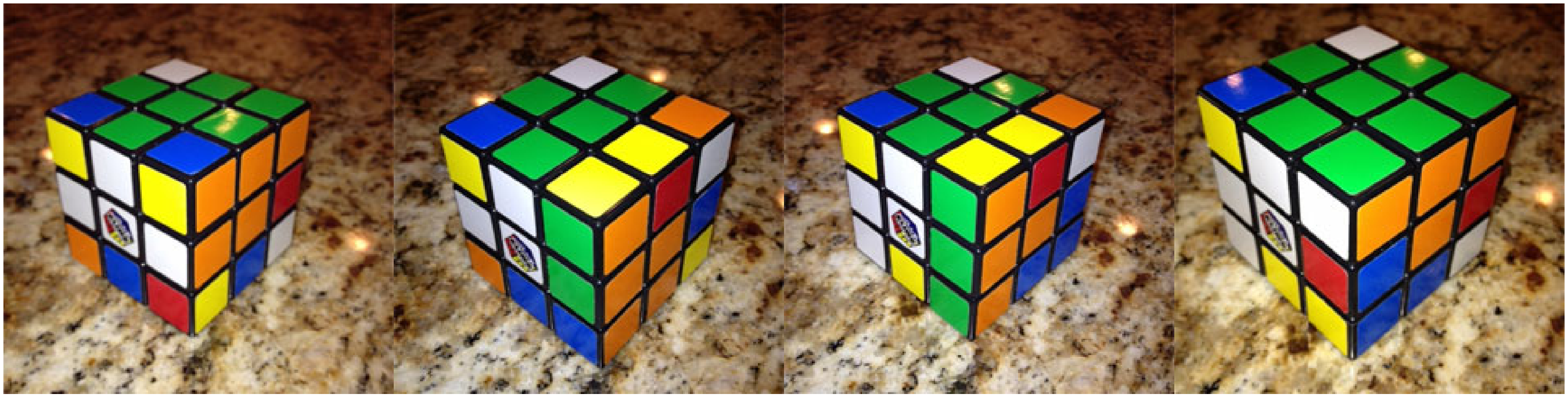
REVISITING THE SOLUTION TO RUBIK'S CUBE

Now that we know a lot about the mathematics behind Rubik's cube, it's time to take a closer at the solution we use. The first part, of course, is pretty easy. When I'm trying to solve Rubik's cube, I always begin with the green center cubelet on top so that I can finish the green face first. I begin with the goal of initially completing the green cross on top, and that is really very easy. I simply rotate faces until I get the green facelet of an edge cube positioned on the down face of the cube. I then rotate the down face until the other color on the edge cube matches the center cube. And finally, I rotate the appropriate face 180° to bring the green facelet to the up face. I then repeat until I've finished my green cross, and I don't really have to formalize the procedure too much since I'm not that worried, yet, about what's going on with the rest of the cube.

Once I've completed the green cross on the up face, I still improvise quite a bit to get the corners positioned. However, I do usually make use of the maneuvers $R^{-1}DR$ and $FD^{-1}F^{-1}$ in order to get my corner cubelet placed with the right orientation. Sometimes, though, the green facelet of a corner cubelet is on the down face of the cube, and when this happens I may do something like $FD^{-1}F^{-1}R^{-1}D^{-1}R$ to rotate it in the bottom layer.



$$R^{-1}DR$$

Once the green face is completed, I turn the cube over so that green is on the down face, and then I proceed in a systematic way to place the edge cubes in the middle layer. The two algorithms that are used in our solution are $URU^{-1}R^{-1}U^{-1}F^{-1}UF$ and $U^{-1}F^{-1}UFURU^{-1}R^{-1}$. What should be clear at this point is that both algorithms are products of commutators which means that they belong to the commutator subgroup of the Rubik's cube group. Also, as we have seen previously, commutators have a tendency to move only a few elements.

Given that, let's look at the first algorithm in a bit more detail. If we perform only the first part, $URU^{-1}R^{-1}$, then the resulting permutation given as a product of cycles can be written as $(UB \quad UR \quad FR)(DRF \quad UFR)(UBL \quad URB)$. We can now see the promise of this permutation. It contains a 3-cycle that involves two edge cubelets on the up face and the edge cubelet in the front-right position. Just what we want!

It also involves a couple of 2-cycles that move corner cubelets, and one of them only switches corner cubelets on the up face. Unfortunately, the other one switches the up-front-right corner cubelet with the down-right-front corner cublet, and that will mess up the green face that we just completed. However, if we perform our algorithm twice, $\left(URU^{-1}R^{-1}\right)^2$, then the result is just the 3-cycle $(FR \quad UR \quad UB)$, and that looks promising except for the fact that this also twists the down-right-front corner cublet into a different orientation.

$$\left(UB \quad UR \quad FR\right)\left(DRF \quad UFR\right)\left(UBL \quad URB\right)$$

Thus, let's see how the commutator $U^{-1}F^{-1}UF$ might fix things for us. If we do this algorithm, then the resulting cycle structure is $(UL \quad UF \quad FR)(DRF \quad UFR)(ULF \quad UBL)$.

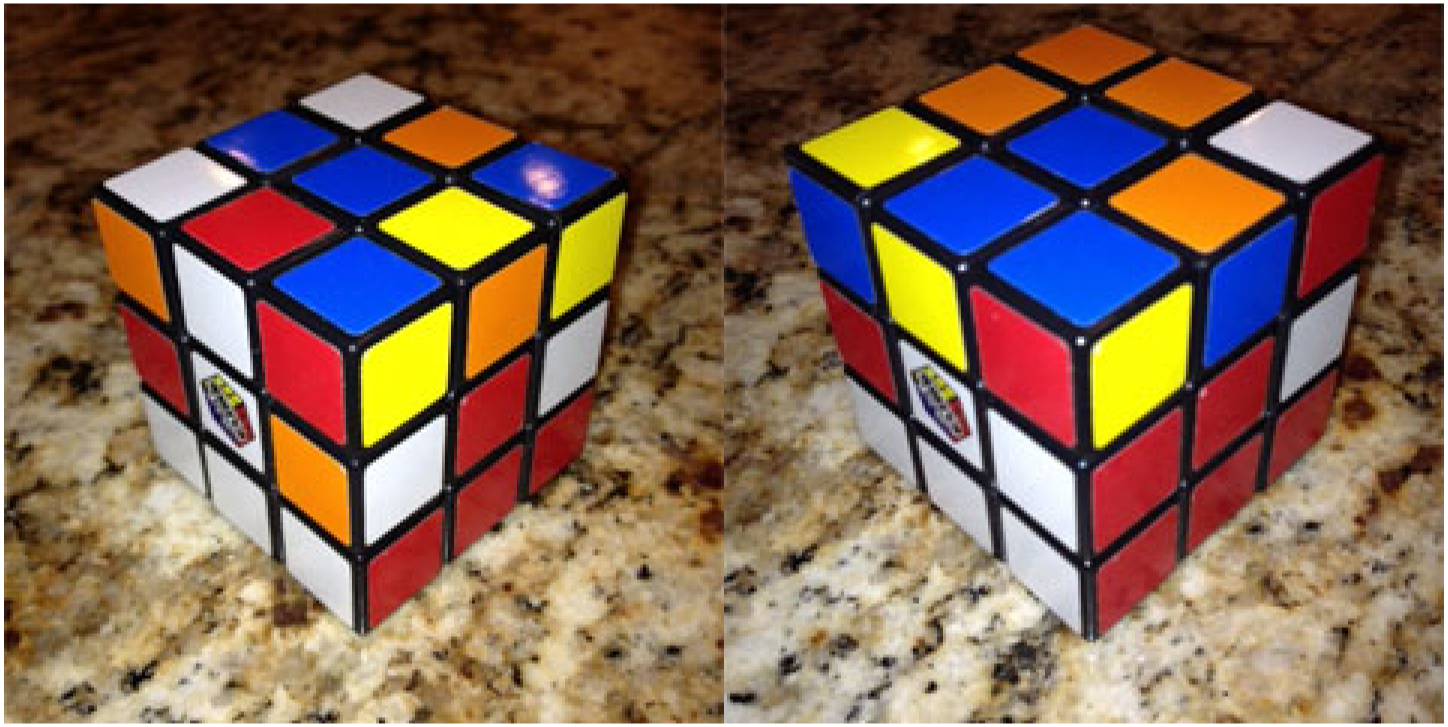
What we immediately see is that our algorithm will once again permute two edge cubelets in the up face with the front-right edge cubelet and it will also switch the down-right-front corner cubelet with the up-front-right corner cubelet. Exactly what we need!

Furthermore, when we multiply the two commutators together we get:

$$(UB \quad UR \quad FR)(DRF \quad UFR)(UBL \quad URB)\,(UL \quad UF \quad FR)(DRF \quad UFR)(ULF \quad UBL)$$

$=(UB \quad UR \quad UL \quad UF \quad FR)(UBL \quad URB \quad ULF)$. From this result we can see that the up-front edge cubelet moves into the front-right position, and everything else that happens is basically a permutation of cubelets on the up face. Absolutely perfect!

Furthermore, when we try it out, we see that the facelet on the front of the up-front edge cubelet remains on the front face when it is moved to the front-right position. This means that our algorithm will work fine just so long as the cubelet that we want to move to the middle layer has the right facelet on the front face, but if its orientation is flipped, then that's why we need our second algorithm, $U^{-1}F^{-1}UFURU^{-1}R^{-1}$. If we look at the cycle structure for this one, then we get $(UL \quad UF \quad UB \quad UR \quad FR)(ULF \quad URB \quad UBL)$. For this one, we need to first get the cubelet we want to place moved into the up-right position, and then our algorithm will move it to the front-right position with the proper orientation.

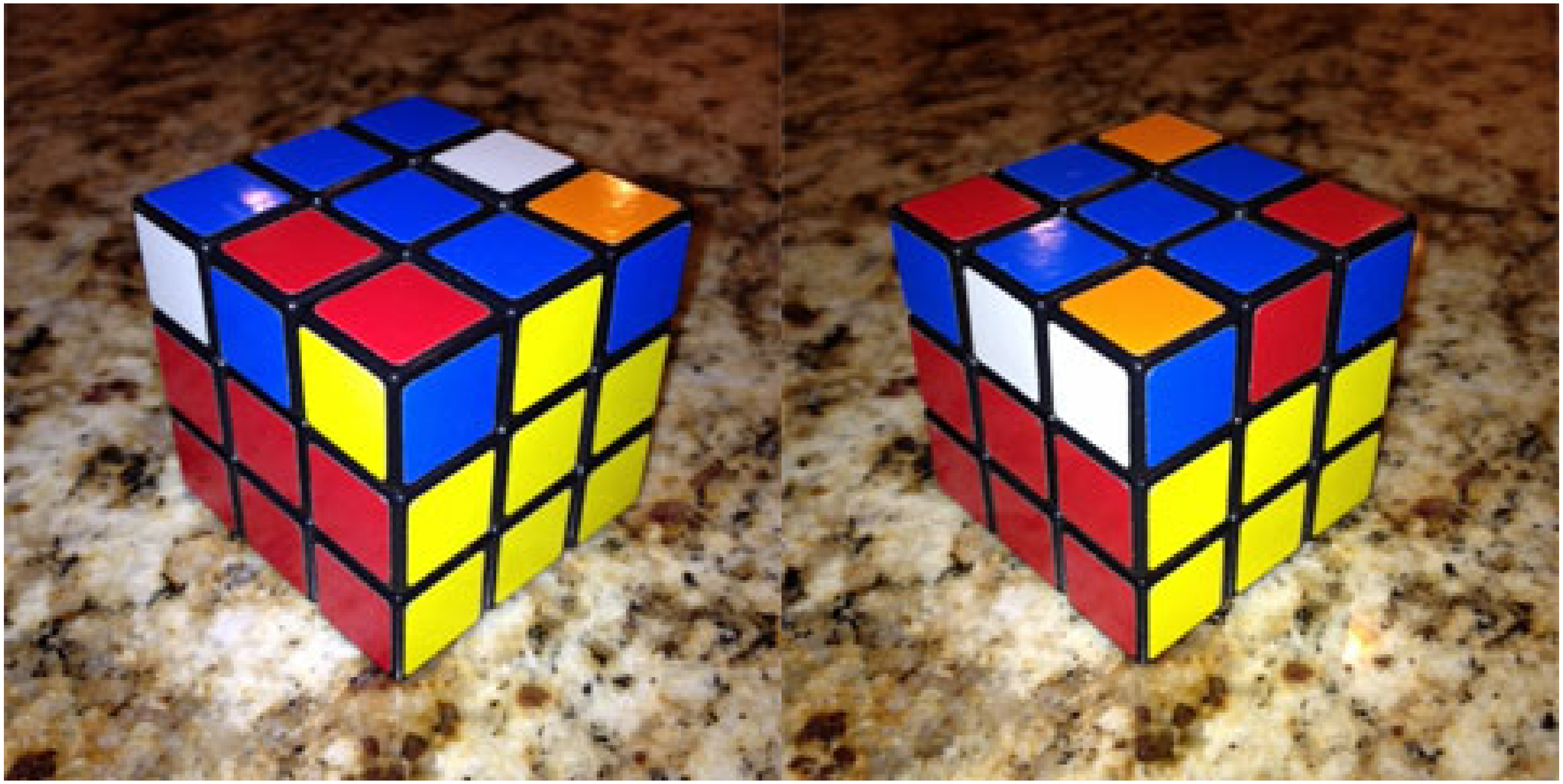


$$URU^{-1}R^{-1}U^{-1}F^{-1}UF$$

The next step in our solution to Rubik's cube is to get the blue facelets on the up face for all of the edge cublets on our top layer, and we can achieve this with the help of the commutator $RUR^{-1}U^{-1}$. However, when we look at the corresponding cycle structure, we see that this commutator moves an edge cubelet on the up face to the front-right position, $(FR \quad UR \quad UB)(DRF \quad UFR)(UBL \quad URB)$. Fortunately, there's an easy fix for this. Simply begin by turning the front face clockwise (F) before doing your commutator, and then turn it back again counterclockwise $\left(F^{-1}\right)$ when you are done. In other words, do $FRUR^{-1}U^{-1}F^{-1}$.

This maneuver will constrain all the movement to the top face, $(UB \quad UF \quad UR)(ULF \quad UFR)(UBL \quad URB)$. When we perform this algorithm, we'll also see that the up-left cubelet never moves, and that two of our edge cubelets get flipped as we go from up-back to up-front and up-front to up-right. Consequently, often all we have to do is to simply repeat this algorithm until all the edge cubelets have the proper facelet on the up face, and if that doesn't work, then you may have to throw in a rotation of the up face in between applications of the algorithm.

Something else to notice is that if we perform this algorithm three times, then the resulting cycle structure is going to be $(ULF \quad UFR)(UBL \quad URB)$. In other words, we switch the two front corner cubelets on top with each other and we also switch the two back corner cubelets. Additionally, when we actually perform this maneuver, we see that the corner cubelets also get rotated in the process. This is a move that could be useful in creating an alternate solution to Rubik's cube.

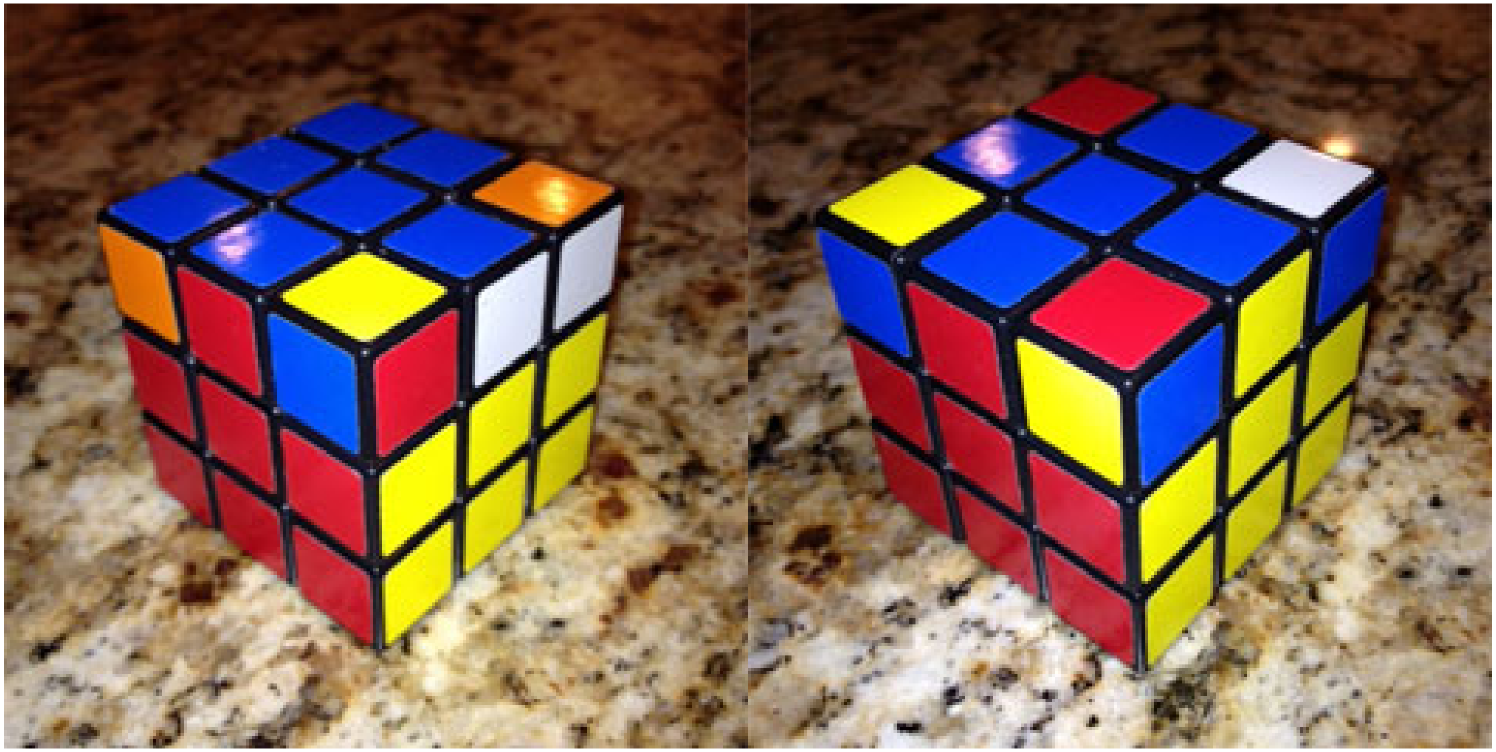


$$FRUR^{-1}U^{-1}F^{-1}$$

Now that we have the edge facelets on the up face properly oriented, we just need to permute the up face edge cubelets until they are all in their proper positions. The algorithm we use for this is $RUR^{-1}URU^2R^{-1}$. Let's break this down a bit. First, perform this algorithm, and keep track of the up-front edge cubelet. What you should notice is that as you do U, U, and U^2, the up-front edge cubelet basically just winds up right back where it started. No change.

However, notice also the presence of the conjugates RUR^{-1} and RU^2R^{-1} in our algorithm. Basically what we are doing with these conjugates is that we are moving an edge cubelet out of the up-right position, rotating the up face, and then moving our edge cublet back into the up-right position, and the end result of $RUR^{-1}URU^2R^{-1}$ is that we permute three of the edge cubelets on the up face with one another. Also, fortunately, nothing below the top layer is disturbed once we complete our $RUR^{-1}URU^2R^{-1}$ algorithm, and the cycle structure for this algorithm is $(UL \quad UR \quad UB)(UBL \quad UFR)(ULF \quad URB)$.

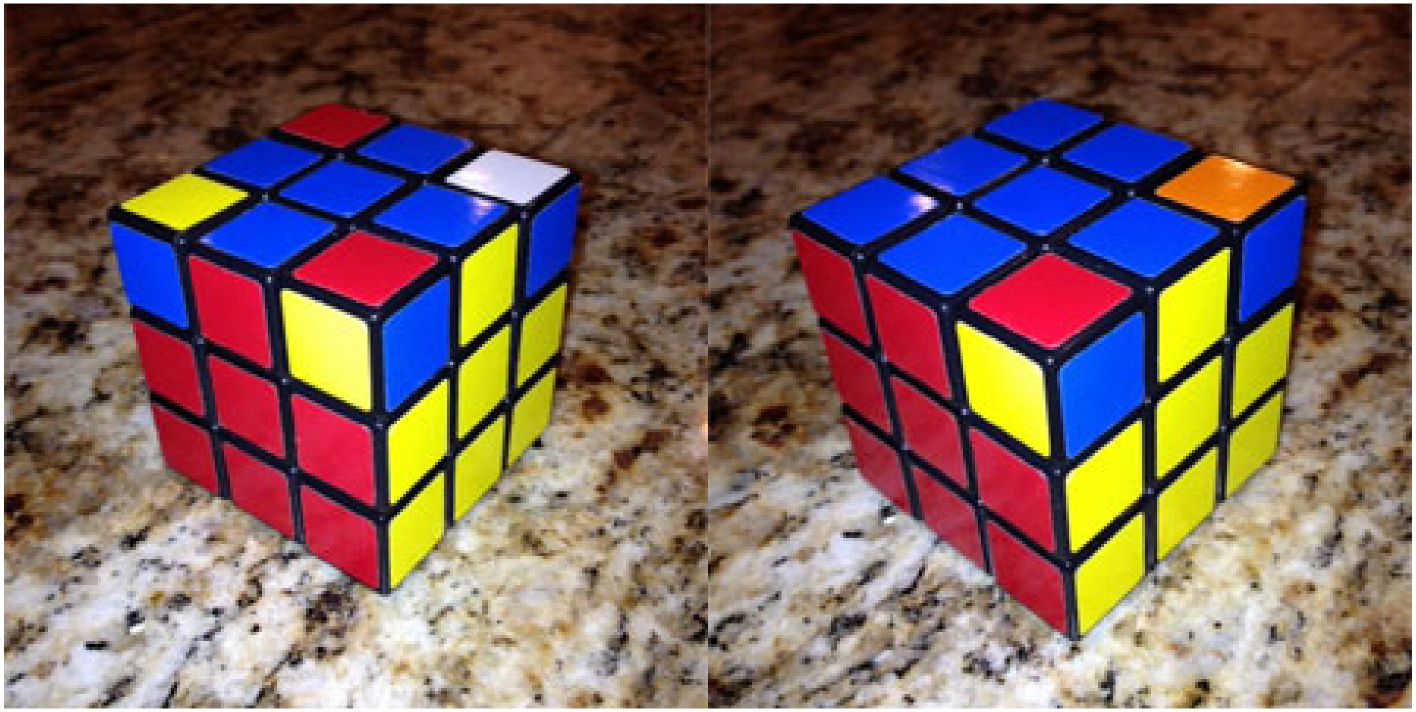
When I apply this algorithm, I usually begin with my red-blue edge cubelet in its proper position, and then I repeat the algorithm until the yellow-blue edge cubelet is properly placed. Then, if I need to, I rotate the whole cube so that I'm looking at the white face, I repeat the algorithm one more time from that position, and then I rotate the up face $90°$ clockwise, and I'm done. Notice, too, that if we did this algorithm three times, then the result would be $(UBL \quad UFR)(ULF \quad URB)$ which means that we are just switching, on the up face, two back corner cubelets diagonally with two front corner cubelets. Again, this, in itself, could be a useful algorithm for an alternate solution to Rubik's cube.



$RUR^{-1}URU^{2}R^{-1}$ done twice

At this point, we just need to permute our corner cubes using the algorithm $URU^{-1}L^{-1}UR^{-1}U^{-1}L$. Embedded in this algorithm we can see the conjugates URU^{-1} and $(L^{-1}U)R^{-1}(U^{-1}L)$. Also, if we removed all the turns of the up face from this algorithm, then we would be left with the commutator $RL^{-1}R^{-1}L$, but by itself this algorithm equals the identity since R and L commute with one another. Hence, we need the rotations of the up face thrown in order to achieve something meaningful. And all that this algorithm wonderfully does is to permute three corner cubelets on the up face, $(ULF \quad URB \quad UBL)$.

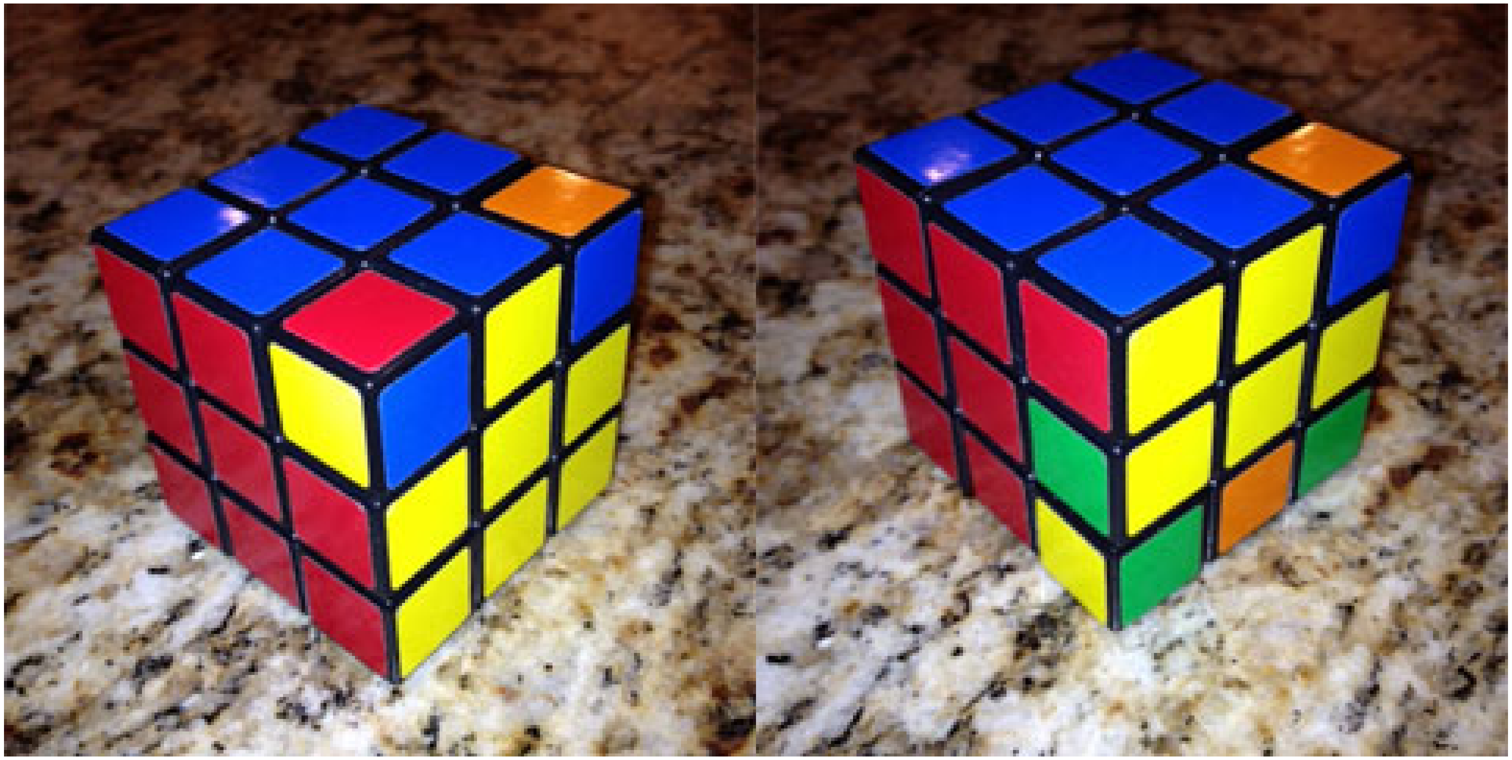
Usually, you want to turn your cube so that the corner cubelet in the up-right-front position is already properly placed. If none of the corner cubelets are in their correct position, then just perform this algorithm once, and you should be able to find one that you can make the up-right-front cubelet just by turning the whole cube. And from there, just keep repeating the algorithm until all the corner cubelets on the up face are in their correct positions. As before, if you perform these algorithms step by step in the software program Rubik, then it's a lot easier to see what's going on!



$URU^{-1}L^{-1}UR^{-1}U^{-1}L$ done twice

Our final move is to simply rotate the corner cubelets on the up face until they get the proper orientation and the cube is solved. To do this, we use the algorithm $\left(R^{-1}D^{-1}RD\right)^2$. Notice that the core of this algorithm is the commutator $R^{-1}D^{-1}RD$, and the cycle structure for this algorithm is $\left(DB \quad DR \quad FR\right)$. The good news is that this algorithm leaves the up-right-front cubelet right where it is, and when we perform it, we also see that it rotates the up-right-front cubelet 120° counterclockwise which is equivalent to a clockwise rotation of 240°. The bad news, of course, is that it messes up the rest of the cube. However, since we have a cycle of length 3, that means that if we perform the algorithm three times, then nothing is left messed up.

Now here's the cool part. Remember that in our earlier chapter on counting the number of permutations in Rubik's cube we saw that every turn of a face would collectively rotate our corner cubelets some multiple of $360°$. Well, that means that when we get down to this final point in solving the cube, our last four corner cubelets are collectively going to be rotated by some multiple of $360°$, and since each application of $\left(R^{-1}D^{-1}RD\right)^2$ results in a rotation of a the up-right-front corner cubelet by $120°$ counterclockwise, the number of times we're going to have to perform this algorithm is going to be some multiple of three, and thus, in the end none of the rest of the cube will be disturbed. Hence, we position a corner cubelet that needs to be rotated in the up-right-front position and apply $\left(R^{-1}D^{-1}RD\right)^2$ until its right, and then we move another corner cubelet on the up face into that position and apply $\left(R^{-1}D^{-1}RD\right)^2$ again until it's correctly oriented. And the end result is that our cube is solved. And it's just that simple!



$$\left(R^{-1}D^{-1}RD\right)^2$$

Rotate the up face and apply $\left(R^{-1}D^{-1}RD\right)^2$ twice

Stick a fork in it, it's done!