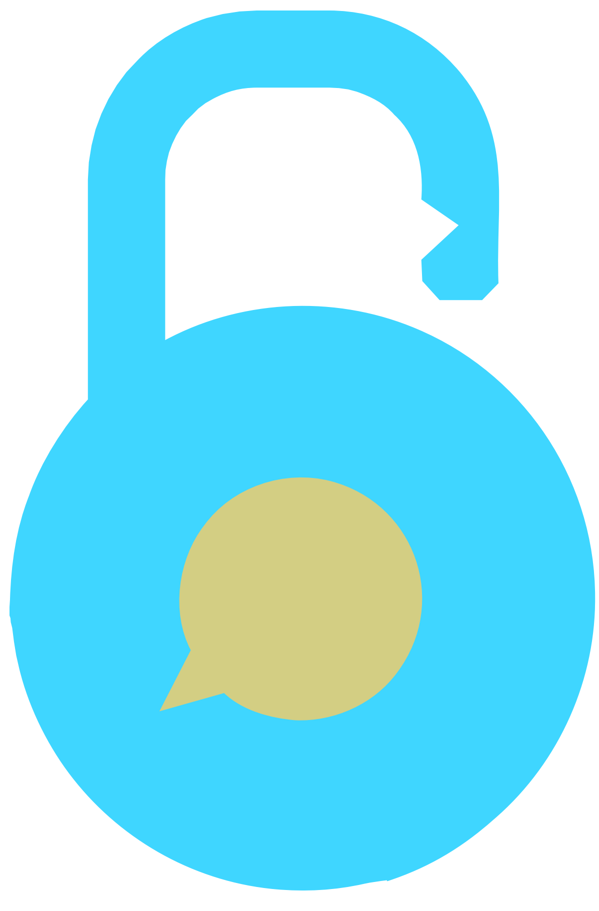
PERMUTATIONS



Definition: A *permutation* of n objects is an arrangement in which order matters. A *combination* is an arrangement in which order doesn't matter.

Example 1: Let's let $A = \{a, b\}$. Then we can write down the elements of this set in two different orders (in other words, two different permutations). We can write either *ab* or *ba*. These represent two different permutations, but the same combination.

Example 2: If you are picking 5 people out of a group of 20 to serve on a committee, then the order in which the people are picked doesn't matter. Hence, we are picking a combination of people.

Example 3: If you are selecting in order 3 people from a committee of 10 such that the first person picked will be the committee chair, the second person will be the vice-chair, and the third person will be the recording secretary, then the order in which the people are picked matters. Hence, we are selecting a permutation of people.

Example 4: If you are dealt a standard 5 card poker hand from a deck of 52 cards, then the order in which you are dealt the cards doesn't matter. Thus, you have been dealt a combination of cards.

Question: Should a combination lock really be called a permutation lock?

There is a ***fundamental counting principle*** that says that if you have a series of choices to make and if you have so many options for each choice, then the total number of possible choices is equal to the product of the number of options at each step along the way. For example, suppose you want to buy a pizza with 1 meat and 1 veggie topping, and suppose you have 3 choices for the meat, 5 for the veggie, 4 choices for the size, and 2 choices for the type of crust. Then to specify your pizza you will have to choose a meat, a veggie, a size, and a crust, and the total number of possible choices is $3 \times 5 \times 4 \times 2 = 120$.

In general, if we want to count the number of permutations of *n* objects that are possible, we simply count the number of ways we can select the first object, the number of ways we can select the second object, and so on. Also, when we make these selections, we are selecting or drawing *without replacement*. That means that once we've selected an object it's not available to be selected again. That's how many things in the world are selected, *without replacement*. The alternative is selecting *with replacement* which means that we can choose the same item over and over again.

Definition: The number of permutations we can make of n objects is $n(n-1)(n-2)\ldots(3)(2)(1) = n!$ (n factorial). Also, by definition we set $0! = 1$. This may seem counterintuitive, but having $0! = 1$ makes our standard counting formulas work out just right.

Example 5: Let's suppose that you have 5 objects and you want to select 3 without replacement. How many permutations are possible?

The number of permutations possible is $5 \cdot 4 \cdot 3 = 60$. We have 5 choices for the first object, 4 for the second, and 3 for the third. Notice that we could also write this in factorial notation as $5 \cdot 4 \cdot 3 = \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 1} = \frac{5!}{2!} = \frac{5!}{(5-3)!}$. More generally, the number of permutations of n objects where we choose r (without replacement) is ${}_nP_r = n(n-1)(n-1)\ldots(n-r+1) = \frac{n!}{(n-r)!}$. Notice, too, that if we ask how many permutations of 5 objects there are if we choose all 5, then the answer is ${}_5P_5 = \frac{5!}{(5-5)!} = \frac{5!}{0!} = 5!$. This is why we had to define $0!$ as being equal to 1.

Example 6: This time suppose that you have 5 objects and you want to select 3 without replacement. How many combinations are possible?

Let's suppose that the objects are the letters in the set $A = \{a, b, c, d, e\}$. Then it should be clear that the number of permutations ${}_5P_3$ over counts the number of combinations because, for instance, *abc* and *cab* represent different permutations but the same combination. Hence, what we need to do is to figure out how many permutations we can make from letters like *abc*, and then that will tell us by what factor we've over counted the number of combinations. Fortunately, that's easy to do!

The number of permutations we can make of the letters *abc* is $3 \cdot 2 \cdot 1 = 3! = 6$. We can also easily list each one of these permutations as I've done below.

$$\begin{matrix} abc & bac & cab \\ acb & bca & cba \end{matrix}$$

We'll denote the number of combinations we can make of 5 objects when we choose 3 as ${}_5C_3$, and according to our discussion above, this should be equal to the number of permutations of 5 objects choose 3 divided by the number of permutations of 3 objects.

In other words, ${}_5C_3 = \frac{{}_5P_3}{3!} = \frac{\frac{5!}{(5-3)!}}{3!} = \frac{5!}{(5-3)!3!} = 10$. More generally, we have that the number of combinations of n objects choose r is ${}_nC_r = \frac{{}_nP_r}{r!} = \frac{n!}{(n-r)!r!}$.

Example 7: Suppose you are ordering a pizza and you are going to select 3 different meat toppings out of 5. How many possibilities are there?

The answer is given to us immediately by ${}_5C_3 = \dfrac{5!}{(5-3)!3!} = 10$. In this case, the order in which you select the meat toppings doesn't matter.

Suppose $P = \{1, 2, 3\}$. Then by a permutation of the numbers 1, 2, & 3 we mean a *bijective* function $f : P \to P$.

What do we mean by a bijective function? Well, this means a function that is one-to-one and onto. The term one-to-one basically means that different elements of the domain always get paired with different elements in the range. A one-to-one function is also called *injective*.

Furthermore, if $f: D \to C$ is a function with *f onto*, then that means that for every $y \in C$, there exists $x \in D$ such that $f(x) = y$. In other words, the range of our function is all of *C* Also, in notation such as the above, *C* is called the *codomain*. Thus, a function is onto if its range is equal to its codomain. When this happens, we also say that the function is *surjective*.

So, having gone through all of that we can say that if $P = \{1, 2, 3\}$, then a permutation of the elements of P corresponds to a bijective function $f : P \to P$. One such permutation could be defined by $f(1) = 2$, $f(2) = 3$, and $f(3) = 1$. However, this may not be the easiest way to visualize the permutation, and so let's explore some other notations.

A notation that is a little handier is the following (along with some variations).

$$\begin{matrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ 2 & 3 & 1 \end{matrix} = \begin{pmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ 2 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

Another way in which we can specify a permutation is in terms of multiplication by what we call a *permutation matrix*. This is a square matrix of zeroes and ones such that each row and each column contains just a single 1. However, when we represent our given permutation using a matrix, we should think in terms of the number 1 moving to position 2, the number 2 moving to position 3, and the number 3 moving to position 1. If we set it up this way, then the product of two permutations will just correspond to the product of one matrix by another.

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} \begin{matrix} \leftarrow \quad \text{3 goes to position 1} \\ \leftarrow \quad \text{1 goes to position 2} \\ \leftarrow \quad \text{2 goes to position 3} \end{matrix}$$

In the long run, though, one of the most convenient notations for permutations is what is called *cycle notation*. For our permutation above the cycle notation simply looks like:

$$\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$$

This is simply a shorthand way of saying that 1 goes to 2, 2, goes to 3, and 3 goes back to 1.

Now suppose we want a permutation such that $f(1)=2$, $f(2)=1$, and $f(3)=3$. In other words, we are switching 1 and 2, but leaving 3 alone. Then this is how we could represent that permutation in each of our three notations.

$$\begin{pmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ 2 & 1 & 3 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}=\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$$

$$(1 \quad 2)(3)=(1 \quad 2)$$

A cycle involving 3 elements is called a 3-cycle, a cycle that switches 2 elements is called a 2- cycle or *transposition*, and a cycle that leaves an element fixed is called a 1-cycle. In other words,

$$\begin{pmatrix}1 & 2 & 3\end{pmatrix} \text{ is a 3-cycle}$$

$$\begin{pmatrix}1 & 2\end{pmatrix} \text{ is a 2-cycle}$$

$$\begin{pmatrix}3\end{pmatrix} \text{ is a 1-cycle}$$

We can multiply permutations together if by multiplication we mean "one permutation followed by another," and when we do so, the result is another permutation. Most of the time, when we right down a product of permutations, we will proceed in order from left to right. Thus, if we want to begin with our 3-cycle above and follow it by our 2-cycle, then we can write the result in our first notation as:

$$\begin{pmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ 2 & 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ 2 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ 1 & 3 & 2 \end{pmatrix}$$

In the matrix notation, the first row of a column matrix corresponds to position 1, the second row corresponds to position 2, and so on. Thus, when we have a permutation in cycle notation like (1,2,3) doesn't mean that we change a column matrix with entries 1,2,3 into one with entries 2,3,1. Instead, it means that 1 moves to position 2, 2 moves to position 3, and then 3 moves to position 1 as indicated in the matrix product below. Again, if we set things up this way, then we can always do two consecutive permutations by multiplying first by one matrix and then the other, but with the first matrix appearing on the right.

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}\begin{matrix} \leftarrow \text{ 3 goes to position 1} \\ \leftarrow \text{ 1 goes to position 2} \\ \leftarrow \text{ 2 goes to position 3} \end{matrix}$$

Thus, using matrices, the first permutation $\begin{pmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ 2 & 3 & 1 \end{pmatrix}$ would be represented by the matrix $A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$, and the second permutation $\begin{pmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ 2 & 1 & 3 \end{pmatrix}$ would be represented by the matrix $B = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. To get the final permutation $\begin{pmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ 1 & 3 & 2 \end{pmatrix}$, however, we have to write our matrices down in the proper order from right to left, or in other words, as

$$BA\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}.$$

And, thus, we see that when we use this approach for matrices, permutation A followed by permutation B does, indeed, gives us the correct composition.

Now let's look at the product of our permutations using cycle notation, and for clarity, I'll also write down all 1-cycles and enclose our separate permutations in brackets. Also, remember that we start on the left and work our way towards the right.

$$\left[(1\quad 2\quad 3)\right]\left[(1\quad 2)(3)\right]=(1)(2\quad 3)$$

The way to decipher this is as follows, and remember to read $1 \to 2$ as “1 goes to 2.”

$1 \to 2$ is followed by $2 \to 1$, and so $1 \to 1$.

$2 \to 3$ is followed by $3 \to 3$, and so $2 \to 3$.

$3 \to 1$ is followed by $1 \to 2$, and so $3 \to 2$.

Therefore, $(1\quad 2\quad 3)(1\quad 2)=(1)(2\quad 3)=(2\quad 3)$.

Notice that in a permutation such as $(1)(2 \quad 3)$, we have two cycles that are disjoint. That means that the cycles don't have any elements in common, and so they don't move anything in common. Consequently, they commute with each other which means that the order in which we write them down doesn't make any difference.

As a final note, a permutation that merely switches two elements is called a transposition, and it turns out that any permutation can be written as a product of transpositions. However, when we do so, the cycles may not be disjoint and there is often more than one way to do it. For example, consider our 3-cycle from above.

$$\begin{pmatrix}1 & 2 & 3\end{pmatrix} = \begin{pmatrix}1 & 2\end{pmatrix}\begin{pmatrix}1 & 3\end{pmatrix} = \begin{pmatrix}2 & 3\end{pmatrix}\begin{pmatrix}2 & 1\end{pmatrix}$$

While a representation of a permutation as a product of transpositions may not be unique, it does turn out that we will consistently wind up with either an even or an odd number of transpositions, and in this way we can classify any permutation as being either even or odd. For example, $\begin{pmatrix}1 & 2 & 3\end{pmatrix}=\begin{pmatrix}1 & 2\end{pmatrix}\begin{pmatrix}1 & 3\end{pmatrix}$ is an even permutation, but $\begin{pmatrix}1 & 2 & 3 & 4\end{pmatrix}=\begin{pmatrix}1 & 2\end{pmatrix}\begin{pmatrix}1 & 3\end{pmatrix}\begin{pmatrix}1 & 4\end{pmatrix}$ is an odd permutation.