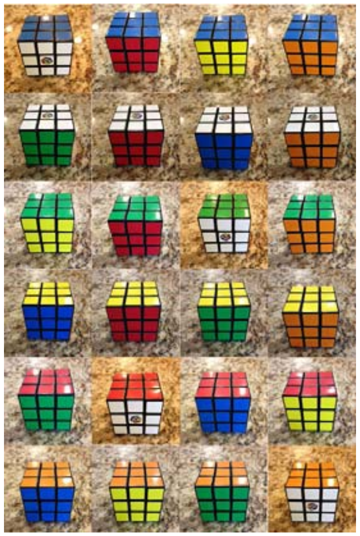
GROUPS OF SMALL ORDER

In this slideshow I just want to do a quick survey of groups of small order. Namely, groups that have orders 1 through 12 plus S_4 which has order 24. Additionally, don't forget that, among other things, S_4 represents the 24 different positions we put a cube in by rotating the whole cube clockwise about the up face, right face, or front face. Remember, also, that we are going to call the group generated by these moves **Benton's Group**. Benton's Group, Benton's Group, Benton's Group! Make me famous!!! ☺



Also, for more information on all of these groups, look up the list of small groups, http://en.wikipedia.org/wiki/List_of_small_groups, in the Wikipedia, or download the free software program Group Explorer by Nathan Carter, http://groupexplorer.sourceforge.net/.

Below, for each order we will list the possible groups up to isomorphism. That essentially means that if two groups are isomorphic, we're pretty much going to call them the same group even if the elements are described differently or if they arise from different contexts. Furthermore, keep in mind that if a group has prime order, then it has to be cyclic since the only subgroups it can have are itself and the identity. If it had any other type of subgroup, then the order of that subgroup would divide the prime which is impossible. Thus, every non-identity element of a cyclic group of prime order generates that group. Also keep in mind that if a group is abelian, then all of its subgroups will automatically be normal subgroups.

Groups of order 1:

There is only one group of order 1, and that's the identity group, the trivial group. Below is its puny little multiplication table. Also, it's abelian.

	e
e	e

Groups of order 2:

Since 2 is a prime number, the only group of order 2 is the cyclic group $\mathbb{Z}_2$, and it's abelian.

Groups of order 3:

Since 3 is a prime number, the only group of order 3 is the cyclic group $\mathbb{Z}_3$, and it's abelian.

Groups of order 4:

There are two groups of order 4 and both are abelian. Thus, from the Fundamental Theorem of Finite Abelian Groups, we know exactly what they are. One must be the cyclic group $\mathbb{Z}_4$, and the other is the direct sum of two cyclic groups, $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ (also frequently written as the direct product $\mathbb{Z}_2 \times \mathbb{Z}_2$). This latter group, by the way, is also called the *Klein four-group* after Felix Klein (1849 – 1925). It is the first group we encounter which is not cyclic.

Groups of order 5:

Since 5 is a prime number, the only group of order 5 is the cyclic group $\mathbb{Z}_5$, and it's abelian.

Groups of order 6:

There are two groups of order 6, one is abelian and the other is nonabelian, and this is also the first time we encounter a nonabelian group in our list. The abelian group is the cyclic group $\mathbb{Z}_6$. You would think that we would also invoke the Fundamental Theorem of Finite Groups and list $\mathbb{Z}_2 \oplus \mathbb{Z}_3$, but it turns out that this group is isomorphic to $\mathbb{Z}_6$ and so we don't need to engage in a needless repetition, $\mathbb{Z}_6 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_3$.

The nonabelian group of order 6 is S_3, the group of the six permutations that we can make of three objects, and it is isomorphic to D_3, the dihedral group that describes the symmetries of an equilateral triangle ($S_3 \cong D_3$).

Groups of order 7:

Since 7 is a prime number, the only group of order 7 is the cyclic group $\mathbb{Z}_7$, and it's abelian.

Groups of order 8:

There are several groups of order 8, three are abelian and two are nonabelian. As usual, it's not too hard to identify the abelian groups, thanks to the Fundamental Theorem of Finite Abelian Groups. We've got the cyclic group of order 8 ($\mathbb{Z}_8$), $\mathbb{Z}_2 \oplus \mathbb{Z}_4$, and $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$. As for the two nonabelian groups, we should be able to immediately guess that one of them will be the dihedral group of the symmetries of a square, D_4. The last nonabelian group of order 8, however, is going to be something completely new to us It's called the *quaternion group*, and its multiplication table is given below.

	1	-1	i	$-i$	j	$-j$	k	$-k$
1	1	-1	i	$-i$	j	$-j$	k	$-k$
-1	-1	1	$-i$	i	$-j$	j	$-k$	k
i	i	$-i$	-1	1	k	$-k$	$-j$	j
$-i$	$-i$	i	1	-1	$-k$	k	j	$-j$
j	j	$-j$	$-k$	k	-1	1	i	$-i$
$-j$	$-j$	j	k	$-k$	1	-1	$-i$	i
k	k	$-k$	j	$-j$	$-i$	i	-1	1
$-k$	$-k$	k	$-j$	j	i	$-i$	1	-1

As you might notice from the multiplication table, quaternions are in some ways like an extension of complex numbers, and in other ways they are like vectors. They were discovered by the mathematician William Rowan Hamilton (1805 – 1865), and they have a history of applications in both math and physics. However, in more recent times they have become an essential part of the mathematics behind the 3D animation that so many people enjoy today. Mathematically, the quaternion group is the first group we encounter in which all the subgroups are normal, but the group itself is nonabelian.

	1	-1	i	$-i$	j	$-j$	k	$-k$
1	1	-1	i	$-i$	j	$-j$	k	$-k$
-1	-1	1	$-i$	i	$-j$	j	$-k$	k
i	i	$-i$	-1	1	k	$-k$	$-j$	j
$-i$	$-i$	i	1	-1	$-k$	k	j	$-j$
j	j	$-j$	$-k$	k	-1	1	i	$-i$
$-j$	$-j$	j	k	$-k$	1	-1	$-i$	i
k	k	$-k$	j	$-j$	$-i$	i	-1	1
$-k$	$-k$	k	$-j$	j	i	$-i$	1	-1

Groups of order 9:

There are two groups of order 9, and they are both abelian. They are the cyclic group $\mathbb{Z}_9$ and direct sum $\mathbb{Z}_3 \oplus \mathbb{Z}_3$.

Groups of order 10:

There are two groups of order 10, and one is abelian and the other nonabelian. One is the cyclic group $\mathbb{Z}_{10} \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_5$, and the other is the dihedral group D_5.

Groups of order 11:

Since 11 is a prime number, the only group of order 11 is the cyclic group $\mathbb{Z}_{11}$, and it's abelian.

Groups of order 12:

There are two abelian groups of order 12 and three nonabelian groups, but the only one we will be interested in the nonabelian dihedral group D_6. This is because our two-squares group, $\langle R^2, U^2 \rangle$, is isomorphic to D_6. This dihedral group has 16 subgroups, the identity, the whole group, seven subgroups of order 2, one subgroup of order 3, three subgroups of order 4, and three subgroups of order 6.

Groups of order 24:

There are fifteen groups of order 24 with three being abelian and the rest nonabelian.

The abelian groups are $\mathbb{Z}_2 \oplus \mathbb{Z}_{12} \cong \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_4 \cong \mathbb{Z}_6 \oplus \mathbb{Z}_4$, $\mathbb{Z}_3 \oplus \mathbb{Z}_8 \cong \mathbb{Z}_{24}$, and $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_6$. The only nonabelian group of order 24 we will be interested in is S_4, the group of all permutations of four objects which is the group we obtain by looking at rotations of the entire cube, i.e. what I like to call **Benton's Group**. This group has thirty different subgroups. Make me famous!