LIFE LESSONS FROM GROUP THEORY AND RUBIK'S CUBE

The main goal of this course has been to introduce you to some of the joys and techniques of group theory and to make traditional, abstract algebra more concrete by relating it to Rubik's cube. However, now that we have covered quite a bit of material, there are some other things I want you to realize. In particular, I hope you recognize that groups are everywhere! For example, anytime you come across cycles or permutations or symmetry in life, there is going to be a corresponding group!

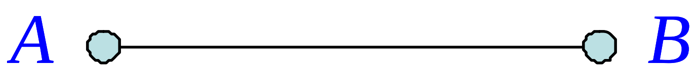
Cycles are something we should see quite easily in our lives. For example, we wake up, we get dressed, we go to work, we come home, we go to sleep, and then the next day this cycle usually repeats itself. This results in a very simple cyclic group that most of us experience on a daily basis. Also, throughout the day we pass by others who are engaged in their own cycles, and if their cycles are disjoint from ours, then they will commute with us, and what one of us does won't affect the other. However, some days we encounter a book or a person or some other situation that brings a new cycle into our lives When we encounter that person, their cycles may not be entirely separate from ours, and then our cycles start to combine with theirs in new and complex ways that are not so independent of one another. And remember that all groups are the product of the ways in which elementary cycles combine and interact with one another.

My life is not only filled with repetitive cycles, it's also replete with symmetry, and by symmetry I simply mean any type of pattern which is repeated. For example, I walk into a classroom, and I see row after row of identical desks. That's symmetry! And corresponding to that symmetry there will be a mathematical group! I look at every human being I encounter, and I see bilateral symmetry that can mathematically be described by the group of integers modulo 2, $\mathbb{Z}_2$. I look at buildings and logos and streets and plants, and everywhere I look I see symmetry. And that means that everywhere I look there are groups that describe the symmetry I am experiencing. There are also patterns of behavior in my life that I repeat, and those behavioral patterns are also a type of symmetry that may be described by groups.

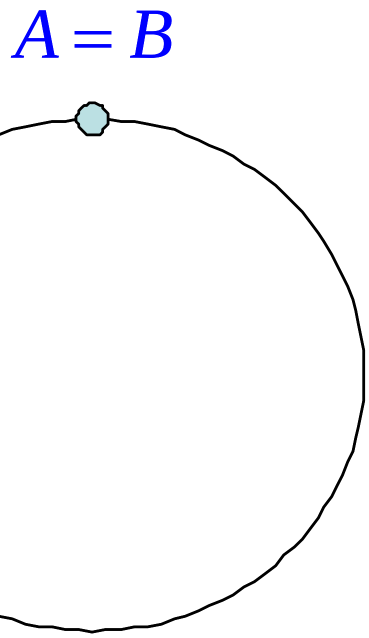
As an example, following this paragraph is a picture of my fireplace. The square shape reminds me of D_4, the symmetries of the square. The eighteen square pieces of stone surrounding the fireplace bring to mind $\mathbb{Z}_{18}$, the cyclic group of order 18, as well as D_4 again. The rectangular wooden beam at the top has two axes of symmetry, one vertical and one horizontal, and associated with this symmetry is $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, the Klein four-group. And finally, the guitar in the picture exhibits bilateral symmetry ($\mathbb{Z}_2$), and the six strings on the guitar suggest both $\mathbb{Z}_6$, the cyclic group of order 6, and S_6, the group of all permutations to be made from six objects. As you can see, we can go on and on identifying symmetries and their corresponding groups! And don't even get me started talking about the symmetries of the curtains and the floor tiles! Groups and symmetry are everywhere!

Permutations also play a large role in our lives. For example, I often say that the only difference between a messy room and a clean room is how things are arranged. Whether we are cleaning our room or mowing the lawn or arranging the furniture, we are just making a different permutation of what's already there. Some philosophers like to refer to these permutations of existing objects as *something from something creation*. In other words, we are just taking things that already exist and creating a different arrangement of them. Also, it's likely that much of your daily life is spent making permutations of one sort or another. For instance, you might spend a lot of your day filing papers, organizing mail, cleaning rooms, doing yard work, or processing orders, and what all of these activities have in common is that they all involve generating permutations of the existing reality. They are all *something from something creation.*

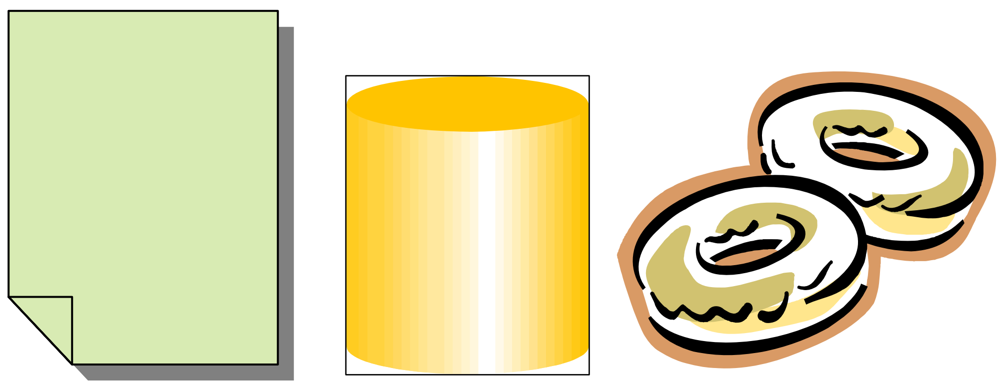
One of the last things I talked about were quotient structures, and I want to talk more about this topic now and expand it beyond the confines of just group theory. The quotient structure is one of those concepts that permeates all of mathematics, and, in a sense, it results in a change in our reality as a consequence of us suddenly seeing things that we previously had considered distinct as now being equivalent. For instance, when we look at the integers, at some point we realize that we can group all the numbers divisible by 2 together, and that creates a new reality with just two objects, *even* and *odd*. To give a geometric example, think of having a piece of string with two ends that we'll call *A* and *B*.



If we now attach the two endpoints together, i.e. make $A = B$, then we create a new object, a circle.

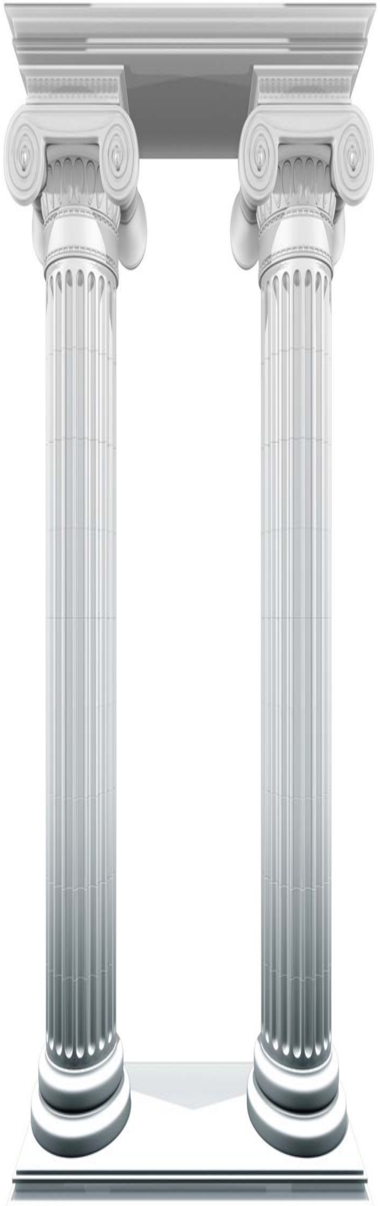


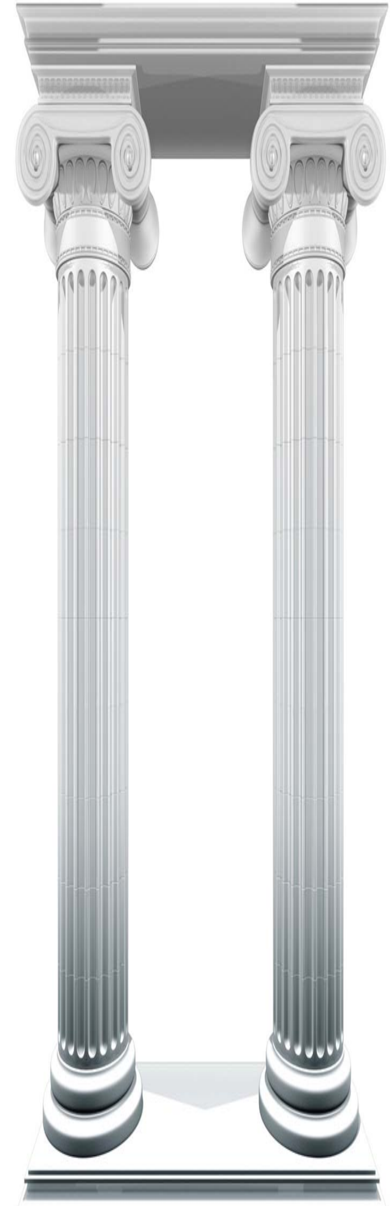
We can think of this new reality as a quotient structure that has been brought about by making *A* equivalent to *B*. In a similar manner, we can transform a piece of paper into a cylinder by making the edges of two opposing sides equivalent to one another, and then we can transform the cylinder into a bagel or donut (what mathematicians call a *torus*) by equating the opposite ends of the cylinder. All of these are examples of creating something new via quotient structures.



I now claim that every time we have an epiphany or insight or “aha!” moment, we are actually creating a new quotient structure for ourselves. For example, you did not always see the world as you do today. You had to learn that the letters “t-h-e” come together to form the word “the.” You had to learn that trees come together to form a “forest.” And you had to learn that various pieces of wood come together to form what you now easily recognize as a “chair.” In other words, at one point in your life, none of these things happened automatically. You literally had to figure out for yourself how objects and ideas somehow combine to form other things, and when you do this, it’s really a quotient structure because components that were previously seen as unrelated now become connected in the new reality, and whatever was previously keeping these things apart has been divided out in the quotient structure. Some would call this *something from nothing creation* because we are literally creating new worlds for ourselves, seemingly out of nothing, by combining things into equivalence classes, and thus, the pieces of wood that at first seemed unrelated now find equivalence in a new concept called “chair.”

Well, so much for grand philosophizing. The point, however, is that cycles, patterns, permutations, symmetry, and groups are all around us, and we now want to go through our days with an increased awareness of them. As we do so, we can also engage in what I call *something from something creation* and *something from nothing creation* by either rearranging our world to make it better or by gaining new insights into things through a process that I see as identical to forming quotient structures. Either way, recognize that groups are everywhere, and live a creative life!





Symmetry is everywhere!