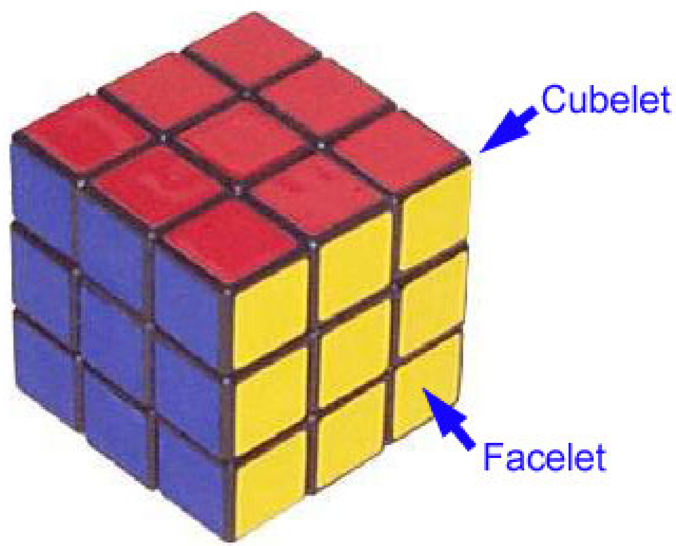
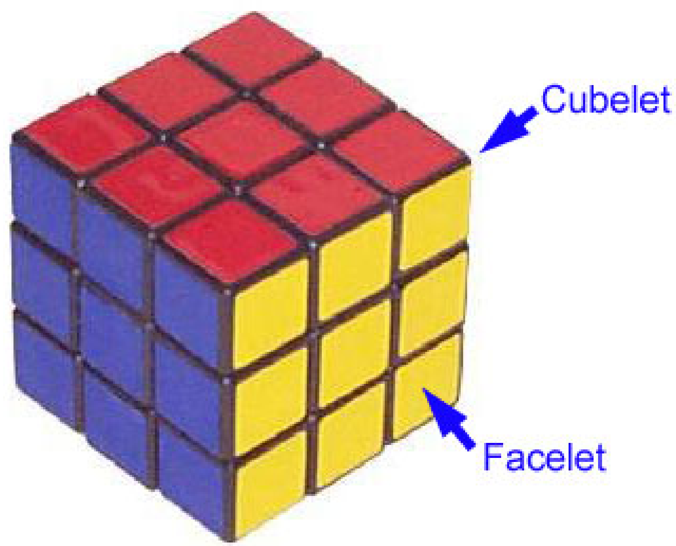
COUNTING THE NUMBER OF PERMUTATIONS IN RUBIK’S CUBE

Rubik's cube is comprised of 54 facelets and 26 cublets.

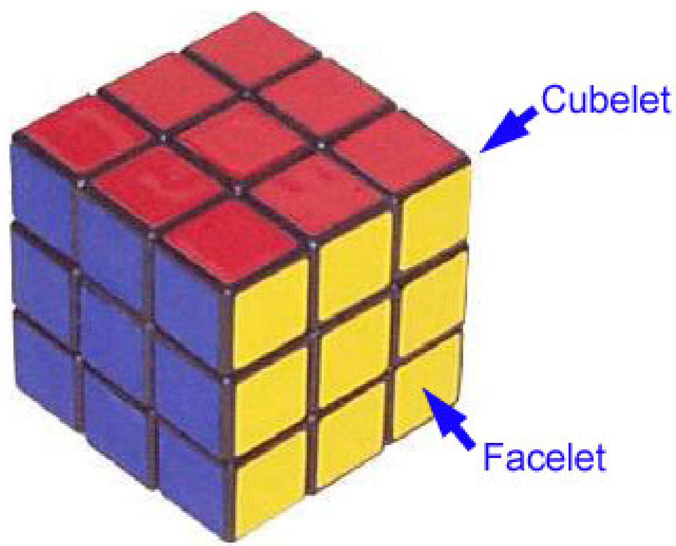


At first glance, you might think that the number of permutations we can make of the 54 facelets is 54-factorial.



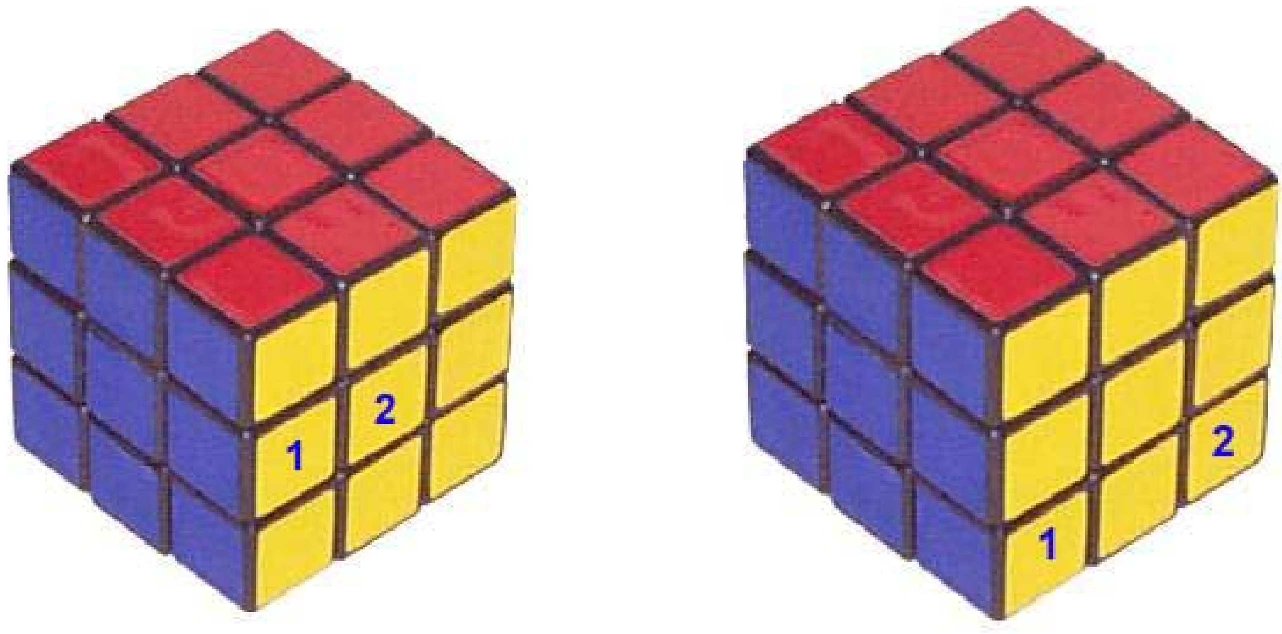
$$54! \approx 2.3 \times 10^{71}$$

However, this number is way too large.



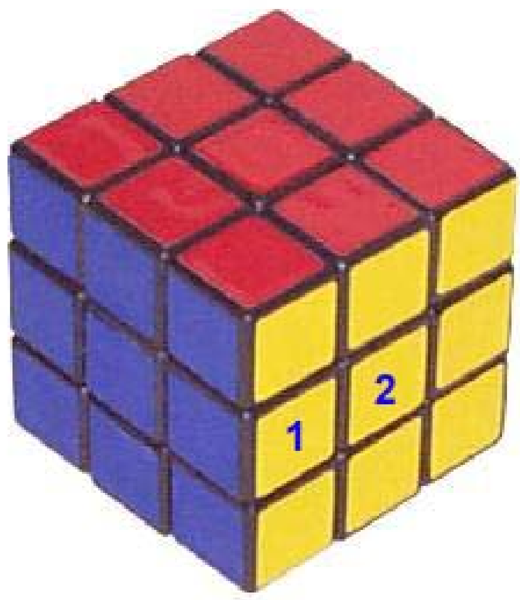
$$54! \approx 2.3 \times 10^{71}$$

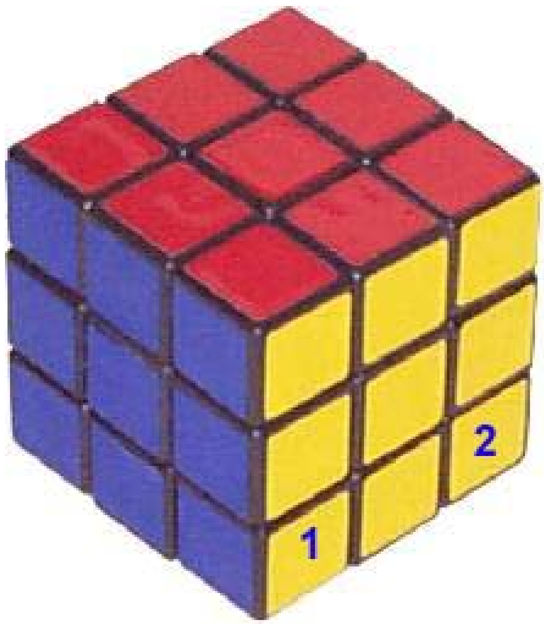
We simply can’t move facelets wherever we would like.



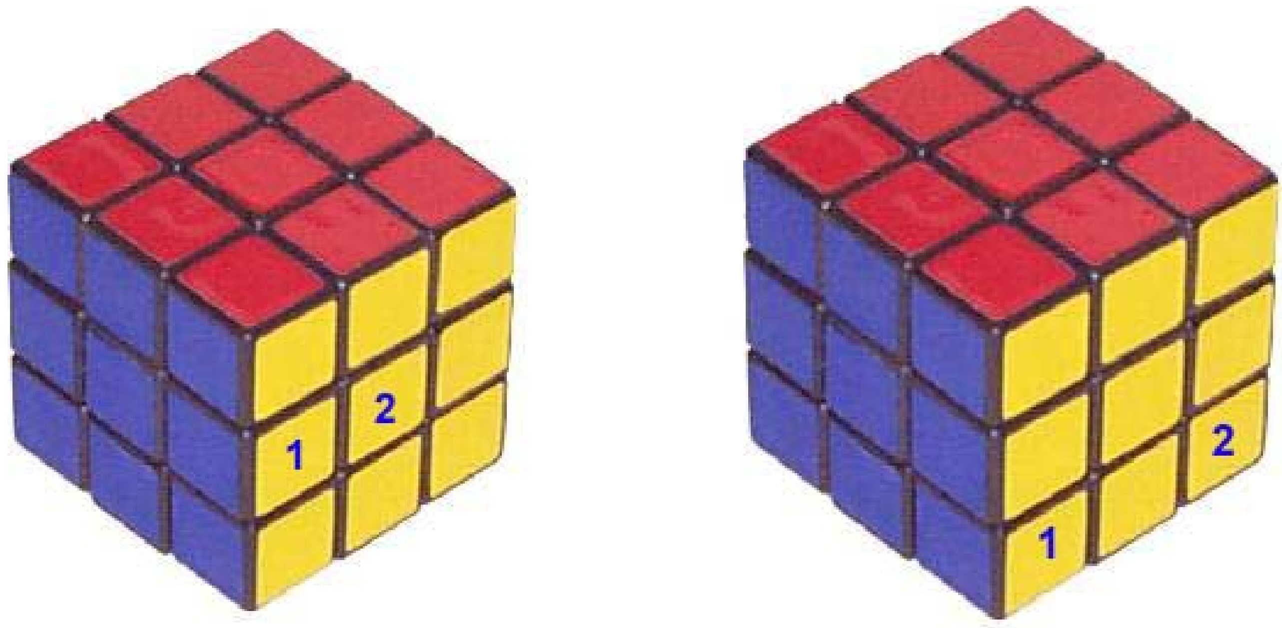
$$54! \approx 2.3 \times 10^{71}$$

For example, we can’t move a facelet on an edge or center cubelet to a corner cubelet.



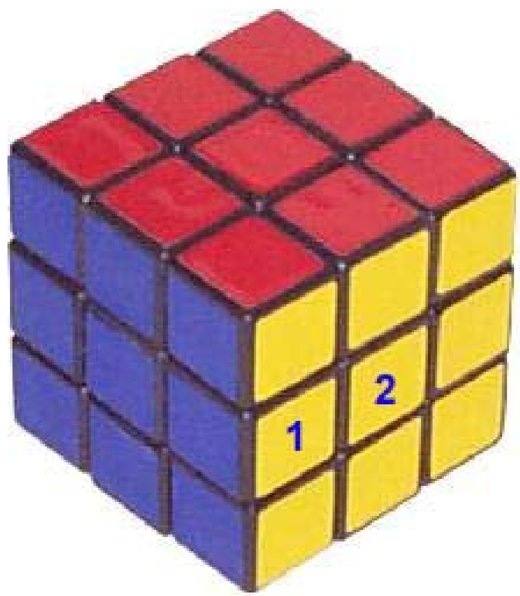


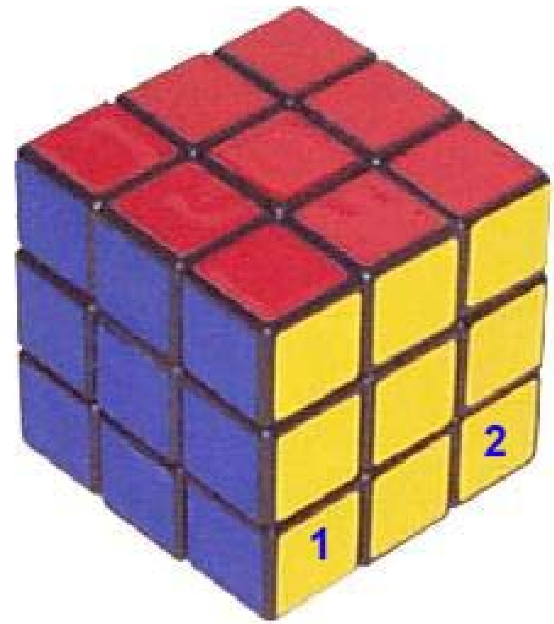
Thus, instead of counting permutations of facelets, perhaps we should count permutations of the 26 cubelets.



$$26! \approx 4.03 \times 10^{26}$$

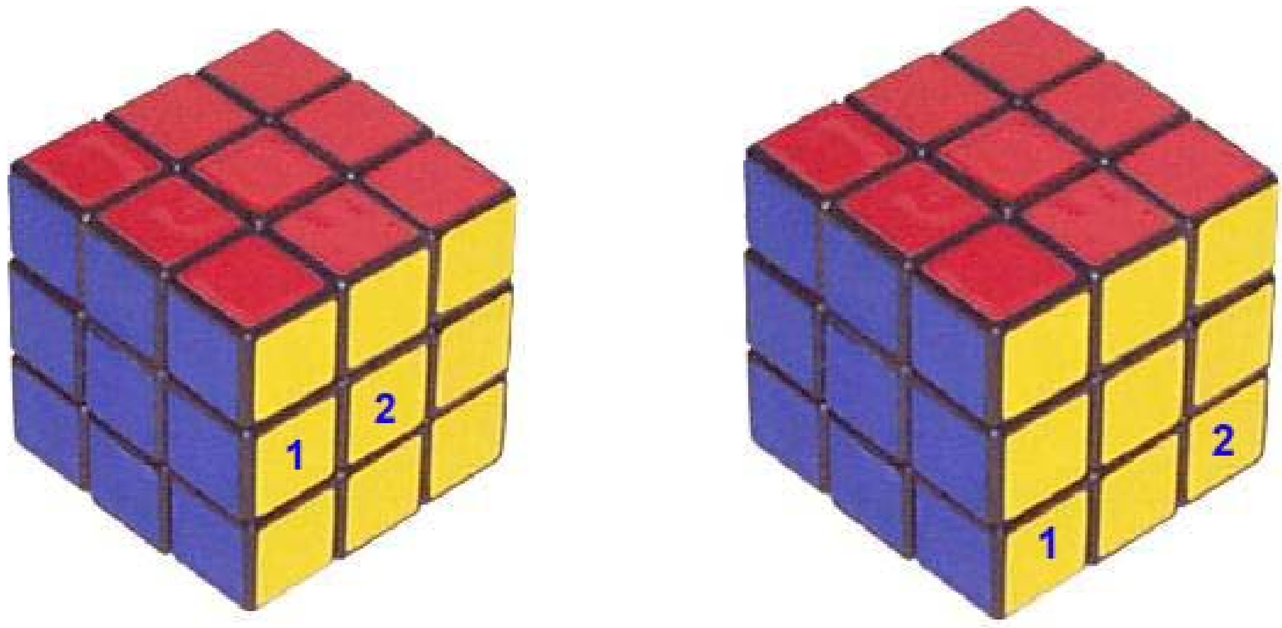
However, this number is still too large.





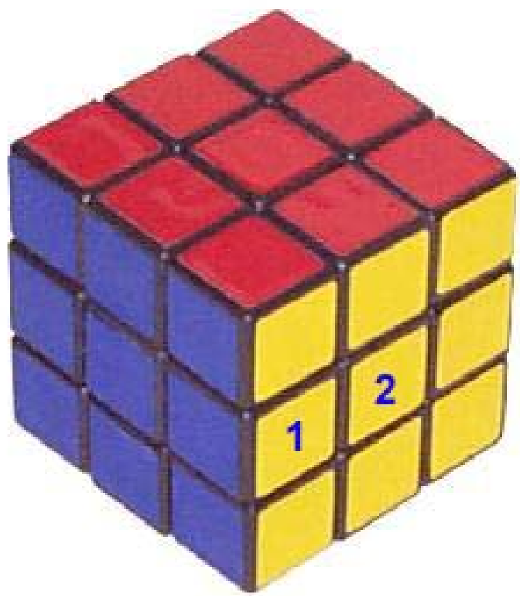
$$26! \approx 4.03 \times 10^{26}$$

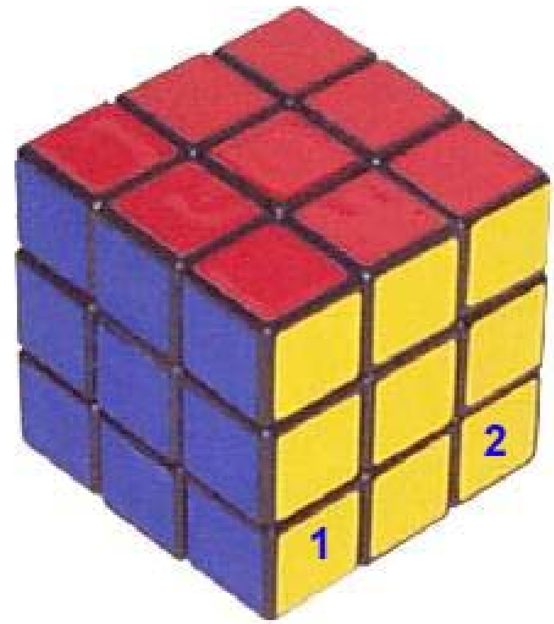
It’s too large because we have three kinds of cubelets – edge, corner, and center cubelets.



$$26! \approx 4.03 \times 10^{26}$$

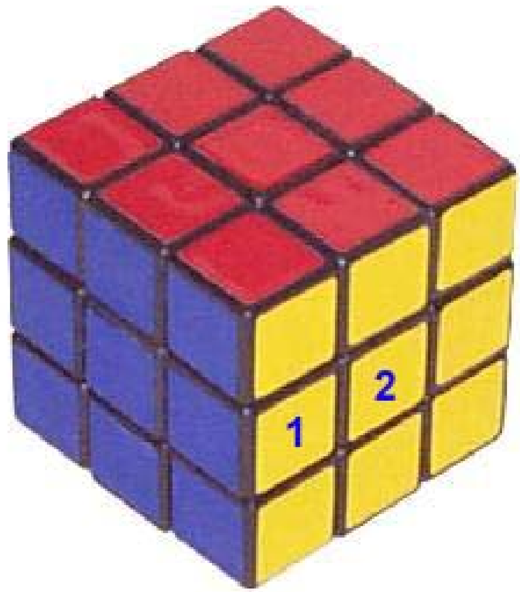
The 8 corner cubelets may be permuted amongst themselves, and the 12 edge cubelets that may also be moved. However, the center cubelets stay where they are.

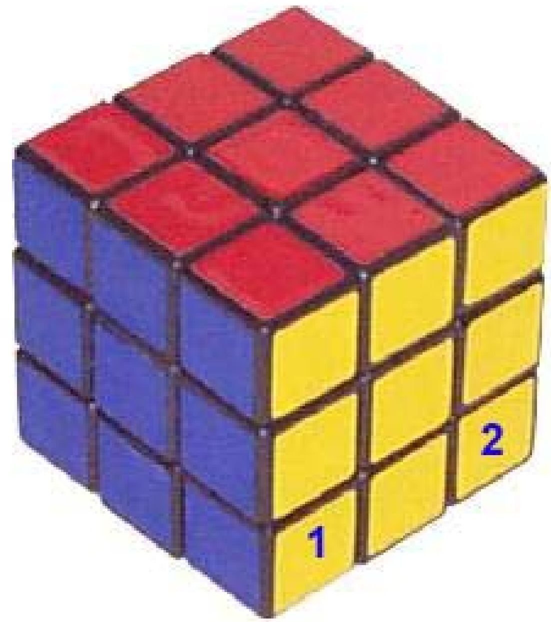




$$26! \approx 4.03 \times 10^{26}$$

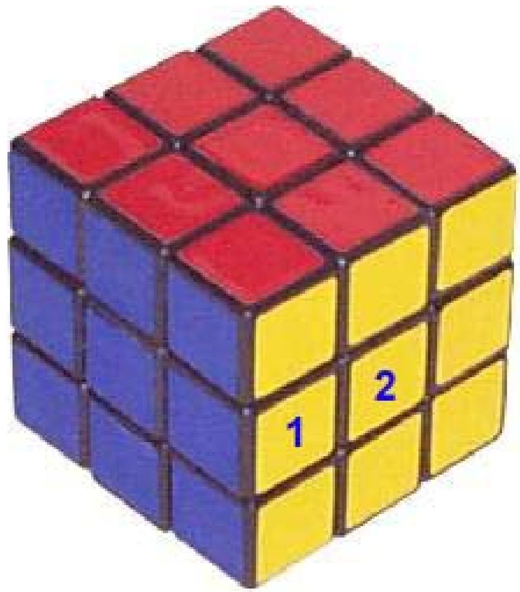
Thus, perhaps the number of possible permutations of the facelets of Rubik's cube is 8-factorial times 12-factorial.

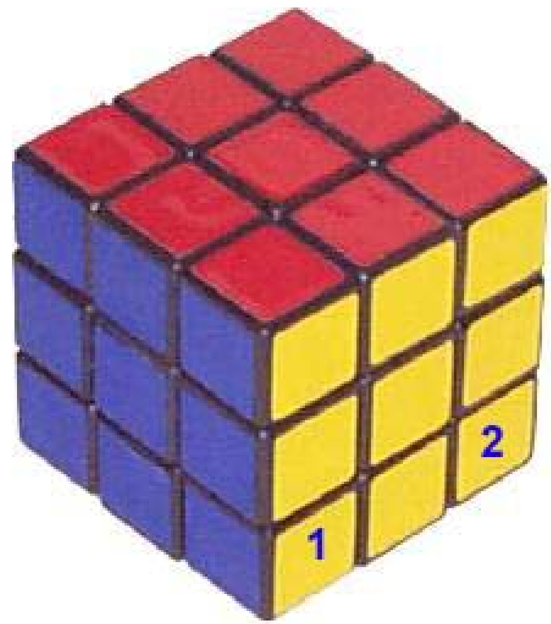




$$(8!)(12!) \approx 1.9 \times 10^{13}$$

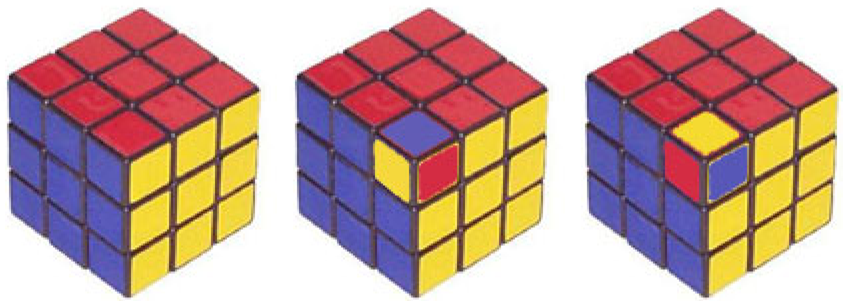
However, this time the count is too low.



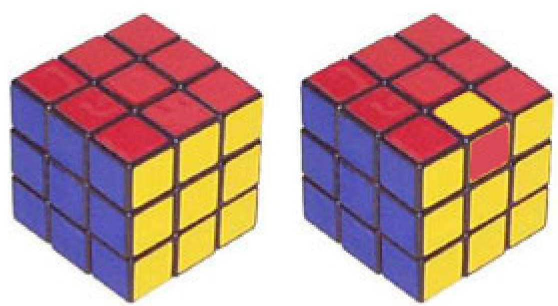


$$(8!)(12!) \approx 1.9 \times 10^{13}$$

It’s too low because each of the 8 corner cubelets can exist in one of 3 different orientations, and each of the 12 edge cubelets can have one of 2 orientations.

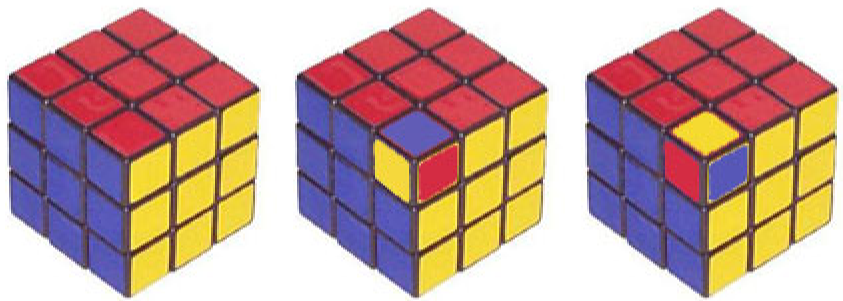


Rotations of a corner cubelet

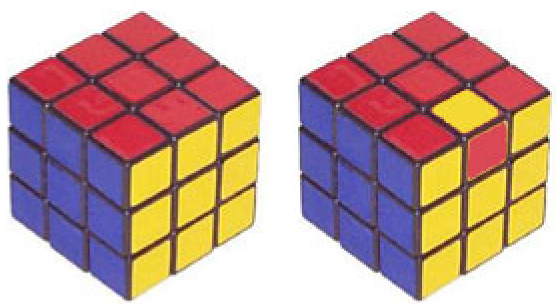


Flipping an edge cubelet

Thus, a better estimate of the number of permutations might be (8!)(12!)(3^8)(2^{12}).



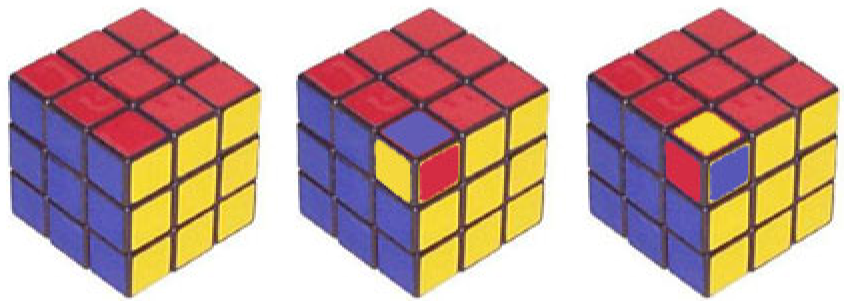
Rotations of a corner cubelet



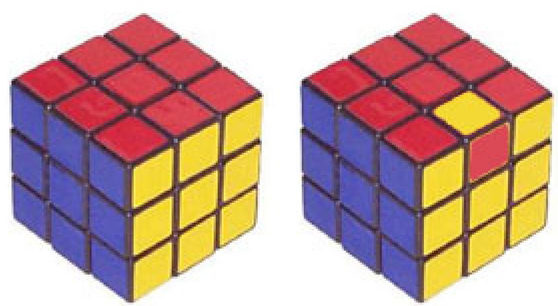
Flipping an edge cubelet

$$(8!)(12!)(3^8)(2^{12}) \approx 5.2 \times 10^{20}$$

However, we are now once again too large.



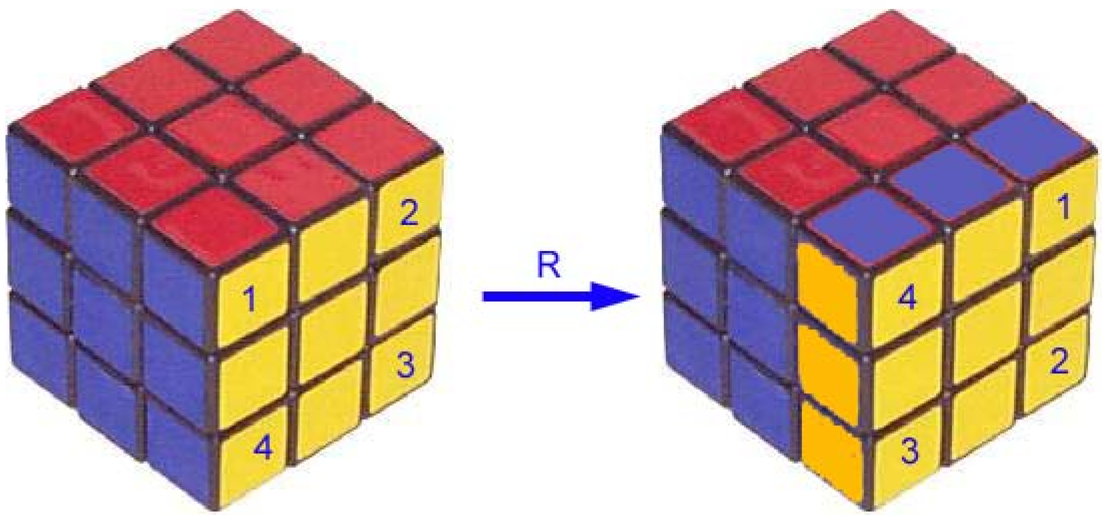
Rotations of a corner cubelet



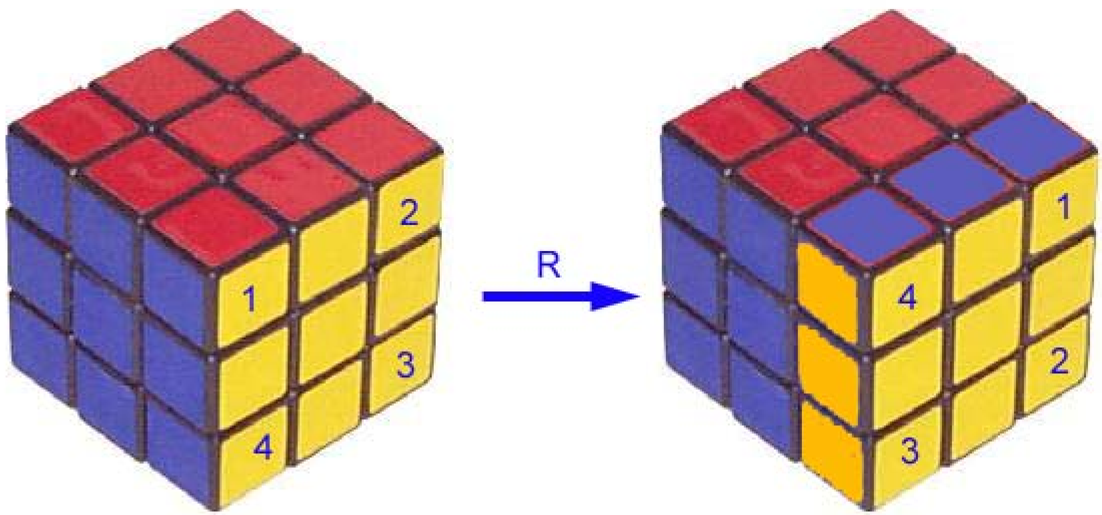
Flipping an edge cubelet

$$(8!)(12!)(3^8)(2^{12}) \approx 5.2\times 10^{20}$$

To see why, we need to express the permutations of our corner and edge cubelets in cycle notation.

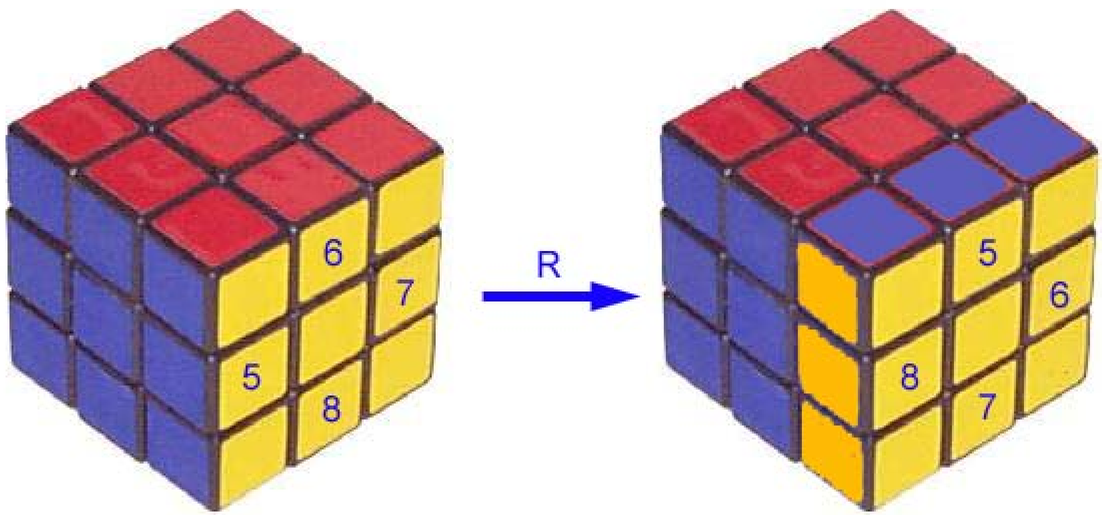


If we rotate the right face of the cube a quarter turn clockwise, then the permutation of the corner cubelets may be written as a product of three transpositions.



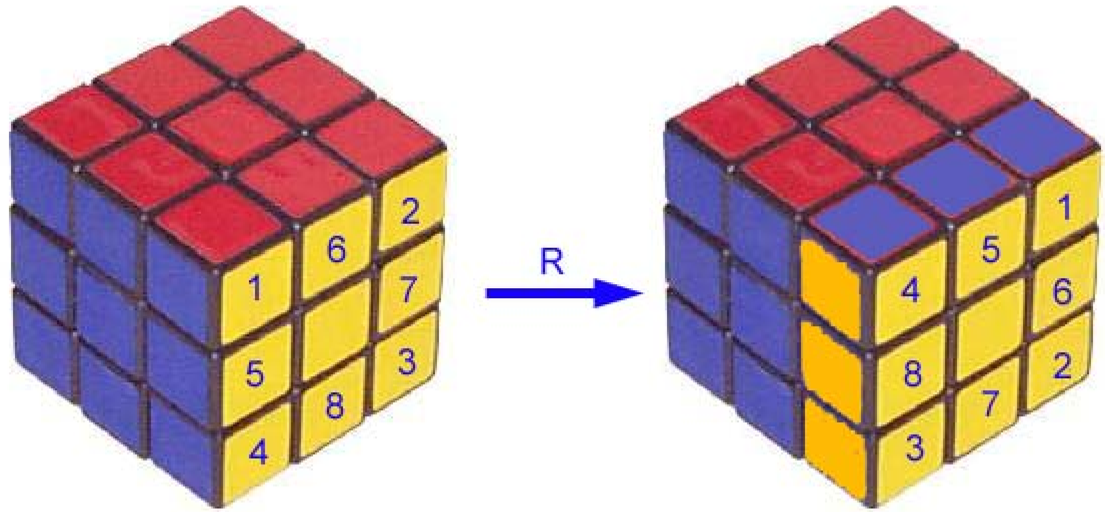
$$\begin{pmatrix}1 & 2 & 3 & 4\end{pmatrix} = \begin{pmatrix}1 & 2\end{pmatrix}\begin{pmatrix}1 & 3\end{pmatrix}\begin{pmatrix}1 & 4\end{pmatrix}$$

Similarly, the permutation of the edge cubelets may also be written as a product of three transpositions.



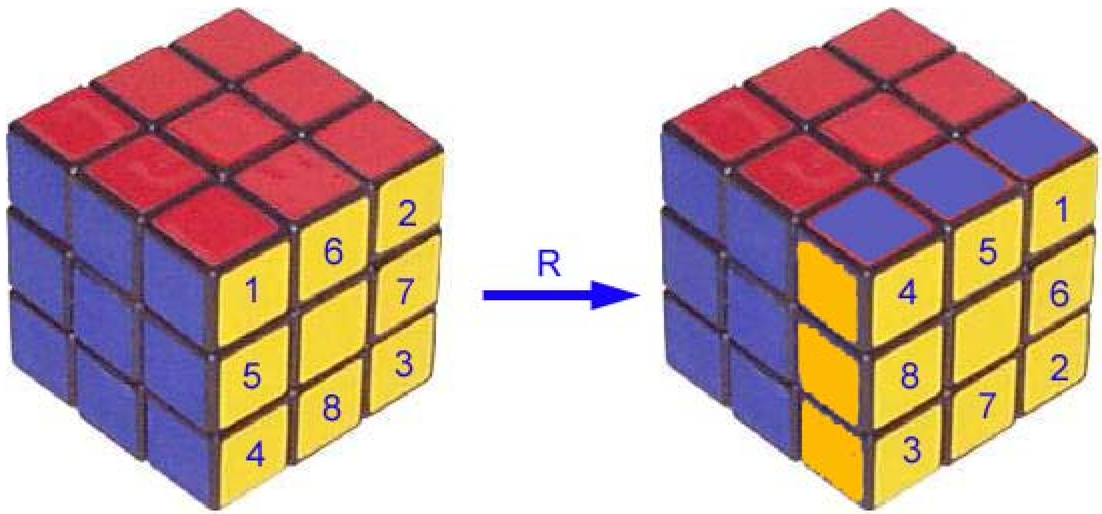
$$\begin{pmatrix}5 & 6 & 7 & 8\end{pmatrix} = \begin{pmatrix}5 & 6\end{pmatrix}\begin{pmatrix}5 & 7\end{pmatrix}\begin{pmatrix}5 & 8\end{pmatrix}$$

Thus, the permutation of all the cubelets on the right face can be written as a product of six transpositions.



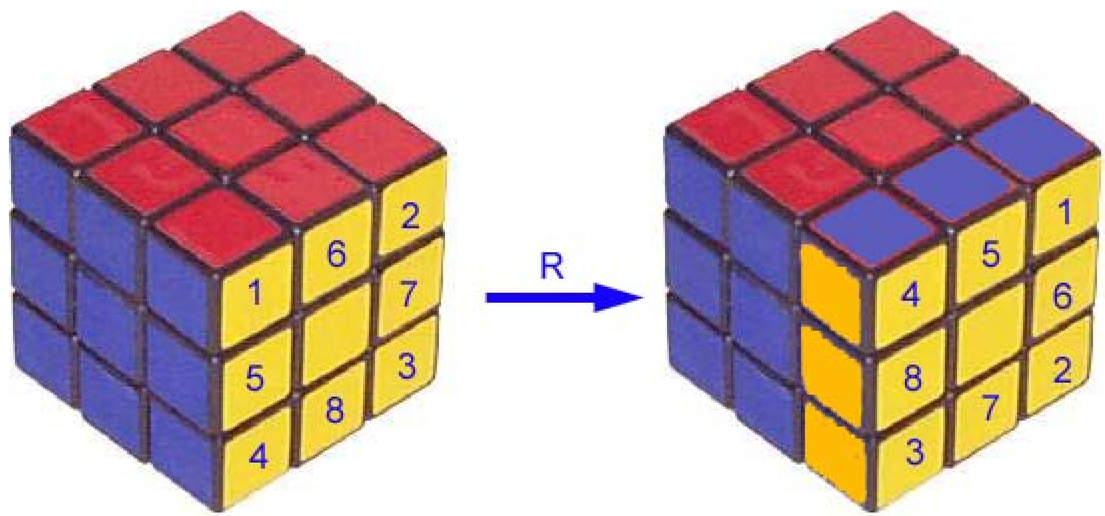
$$(1\quad 2\quad 3\quad 4)(5\quad 6\quad 7\quad 8)=(1\quad 2)(1\quad 3)(1\quad 4)(5\quad 6)(5\quad 7)(5\quad 8)$$

This means that anytime we rotate a face of the cube, we generate an even permutation.



$$(1\quad 2\quad 3\quad 4)(5\quad 6\quad 7\quad 8)=(1\quad 2)(1\quad 3)(1\quad 4)(5\quad 6)(5\quad 7)(5\quad 8)$$

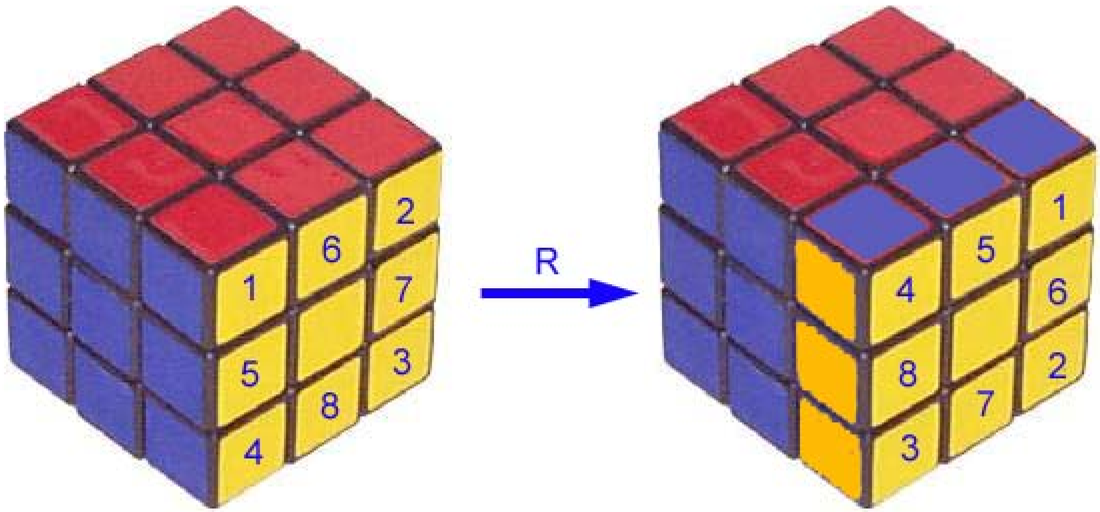
And the even permutations comprise only half of our previous estimate.



$$(1\quad 2\quad 3\quad 4)(5\quad 6\quad 7\quad 8) = (1\quad 2)(1\quad 3)(1\quad 4)(5\quad 6)(5\quad 7)(5\quad 8)$$

$$\frac{(8!)(12!)(3^8)(2^{12})}{2}$$

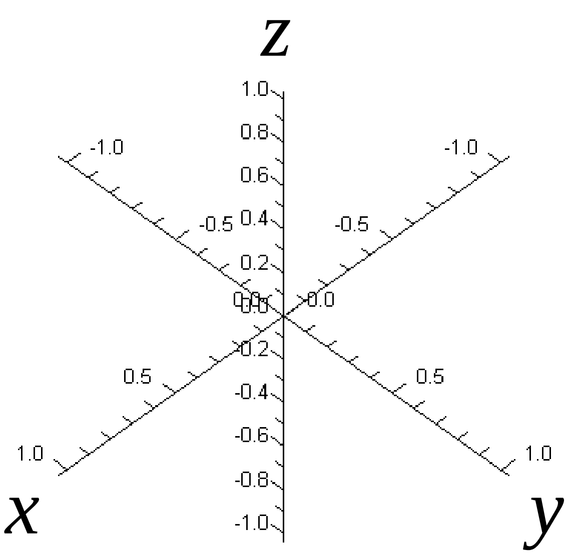
We're getting closer to the correct number of permutations that can be made of the facelets on Rubik's cube, but our estimate is still a little bit too large.

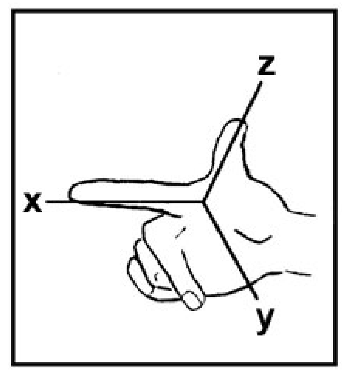


$$\begin{pmatrix}1 & 2 & 3 & 4\end{pmatrix}\begin{pmatrix}5 & 6 & 7 & 8\end{pmatrix} = \begin{pmatrix}1 & 2\end{pmatrix}\begin{pmatrix}1 & 3\end{pmatrix}\begin{pmatrix}1 & 4\end{pmatrix}\begin{pmatrix}5 & 6\end{pmatrix}\begin{pmatrix}5 & 7\end{pmatrix}\begin{pmatrix}5 & 8\end{pmatrix}$$

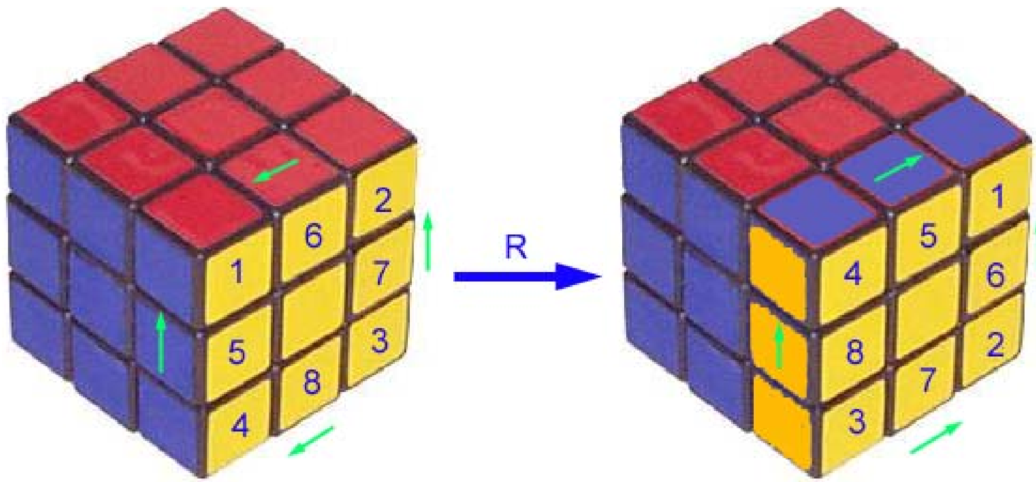
$$\frac{(8!)(12!)(3^8)(2^{12})}{2}$$

To see why, first recall the definition of a right-hand coordinate system and how the fingers and thumb below point in the positive directions of our axes.

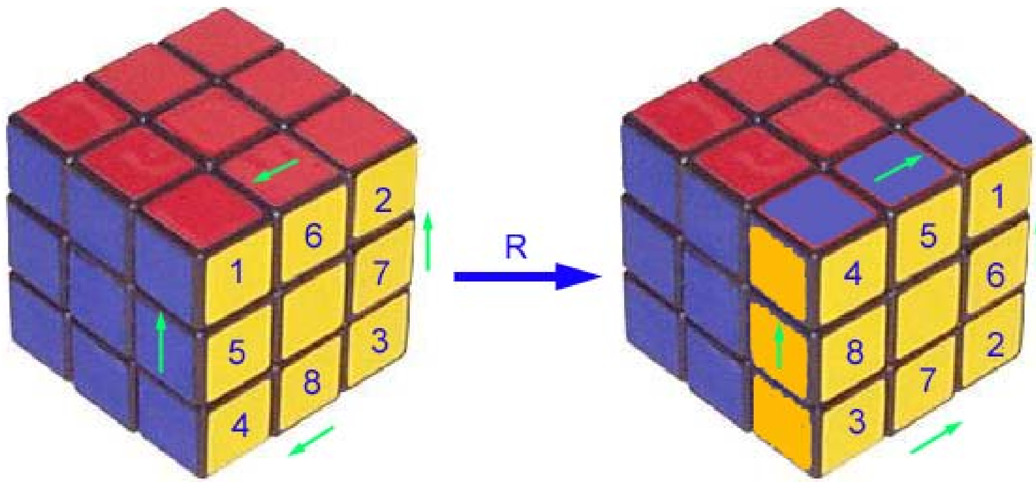




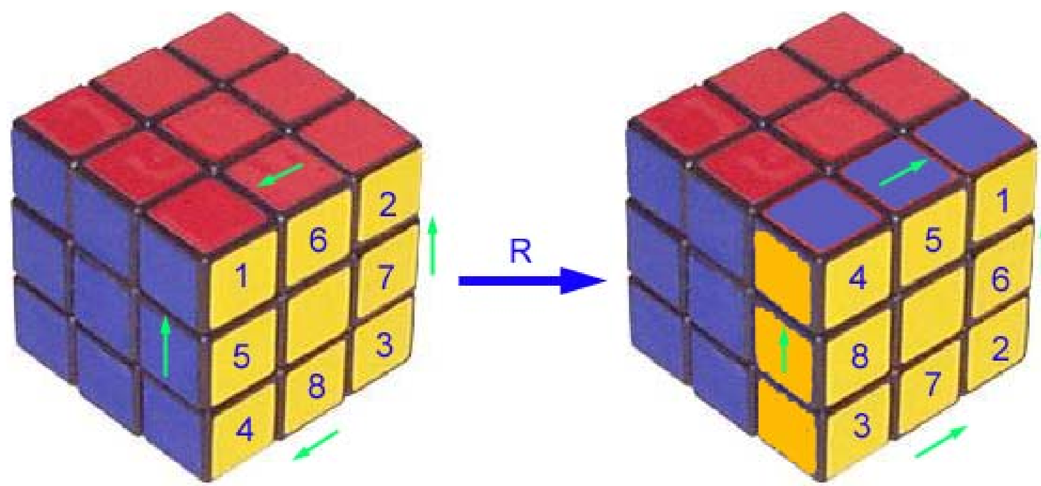
Now let's attach some arrows pointing in the positive directions to the edge cubelets on our right face, and let's rotate that face a quarter turn clockwise.



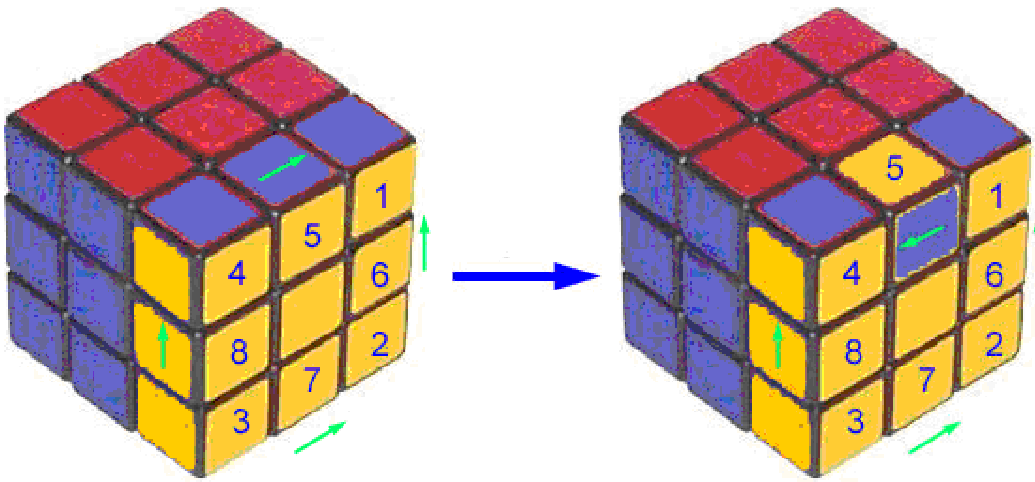
The result is that two arrows reverse direction, but the overall orientation is still (positive)(negative)(positive)(negative)=positive.



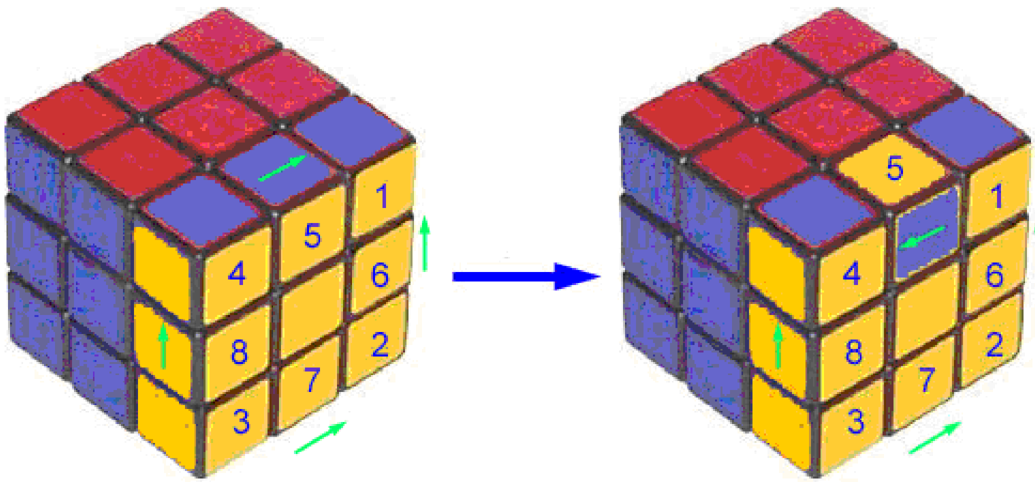
Thus, after every turn of a face on our cube, the edge cubelets are left, collectively, with a positive orientation.



In particular, we can never wind up with a situation like below where a an odd number of edge cubelets have a negative orientation.

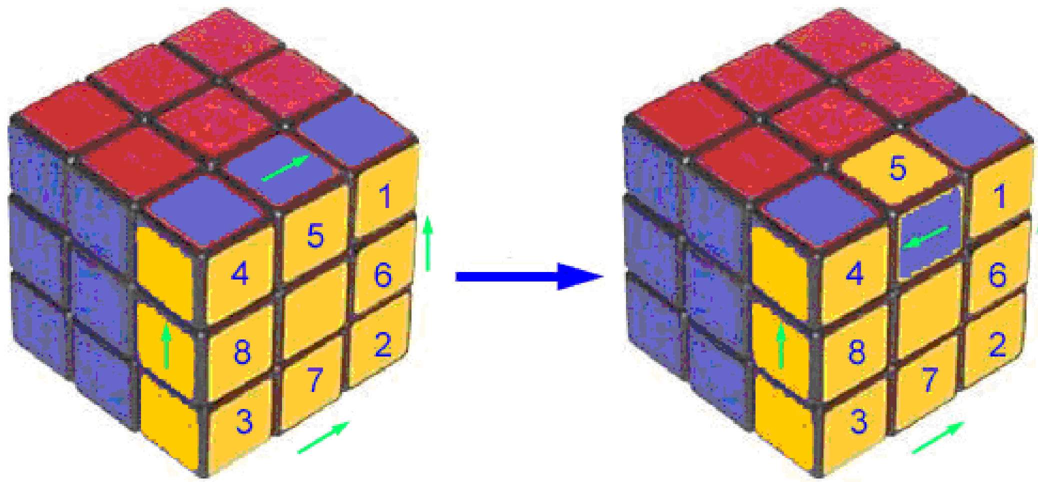


Thus, since we can only wind up, collectively, with positively oriented edge cubelets, we once again need to divide our previous estimate by two.



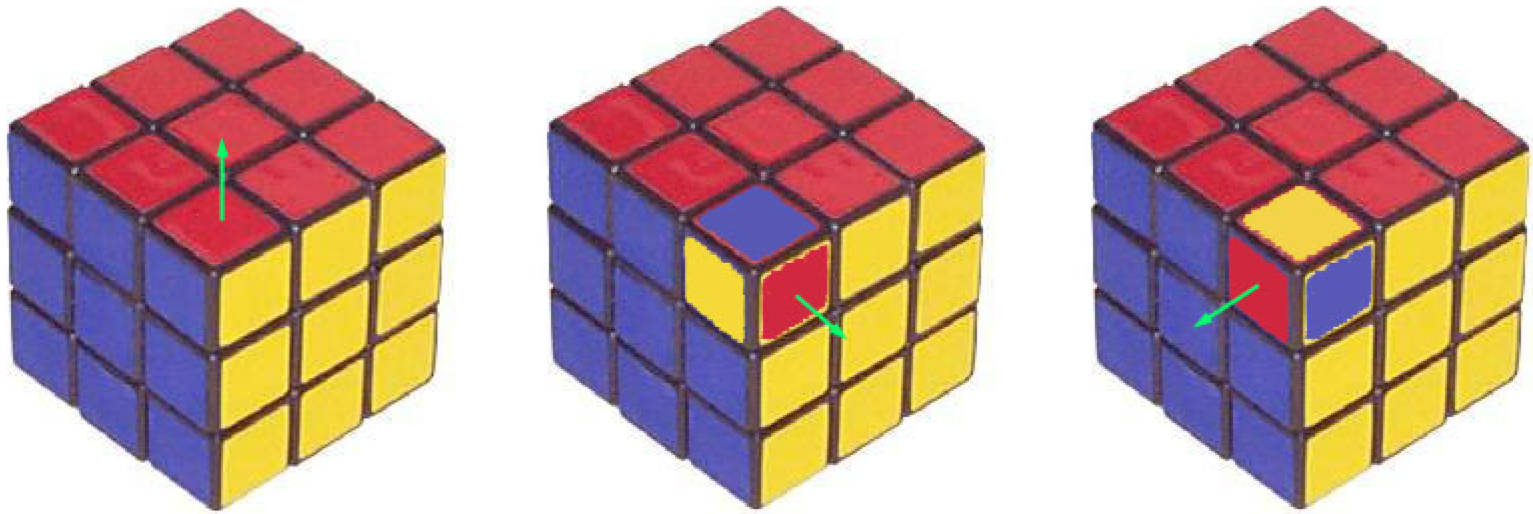
$$\frac{(8!)(12!)(3^8)(2^{12})}{2 \cdot 2} \approx 1.30 \times 10^{20}$$

However, this last estimate is still slightly too large.

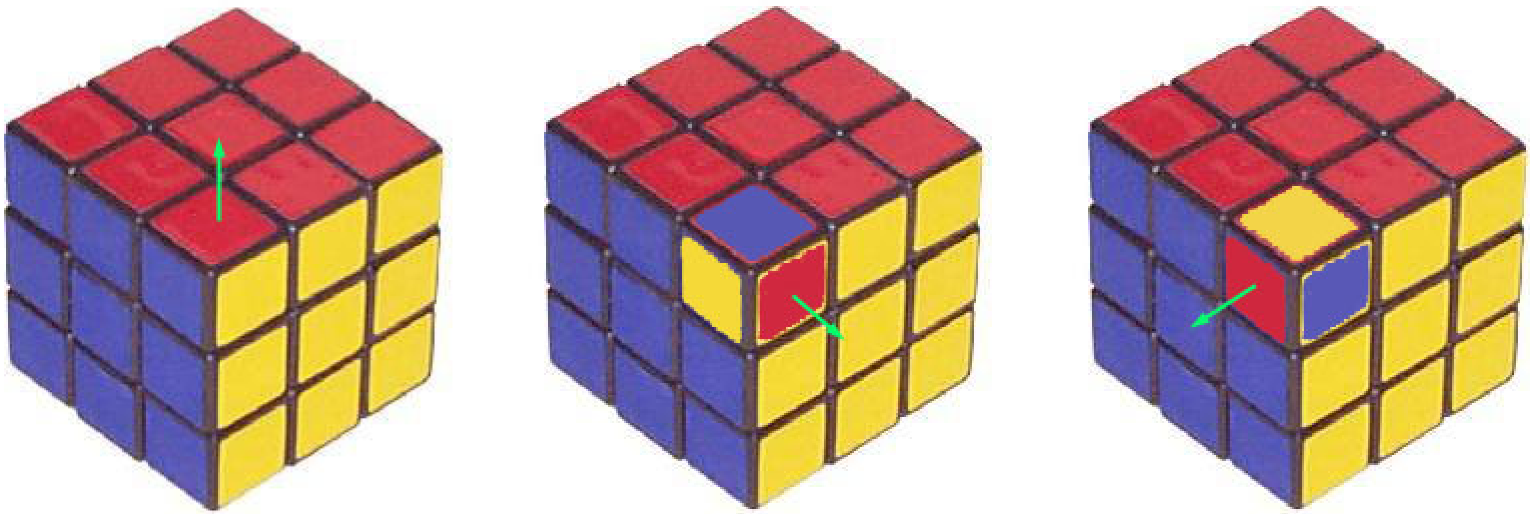


$$\frac{(8!)(12!)(3^8)(2^{12})}{2 \cdot 2} \approx 1.30 \times 10^{20}$$

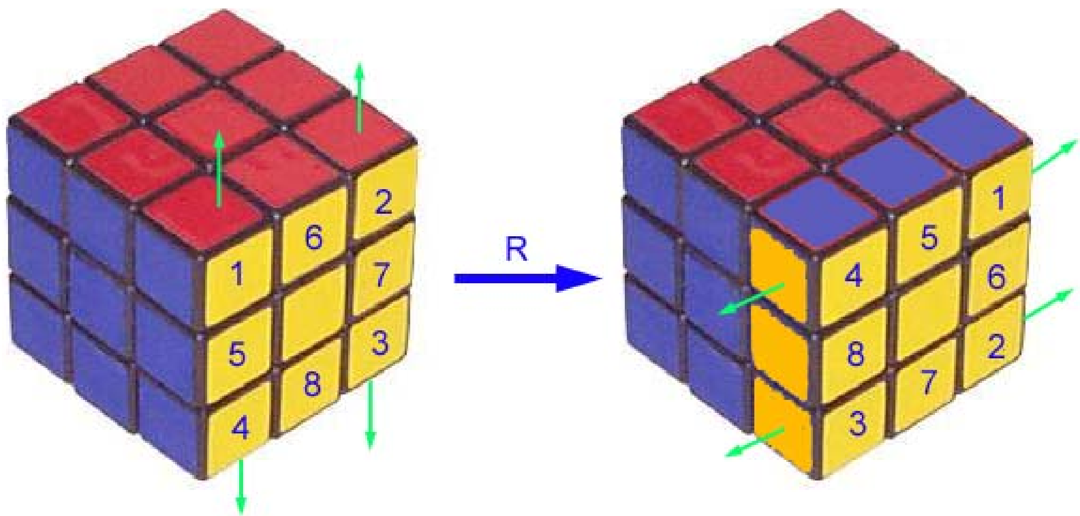
To see why, we need to understand that each corner cubelet can exist in one of three orientations.



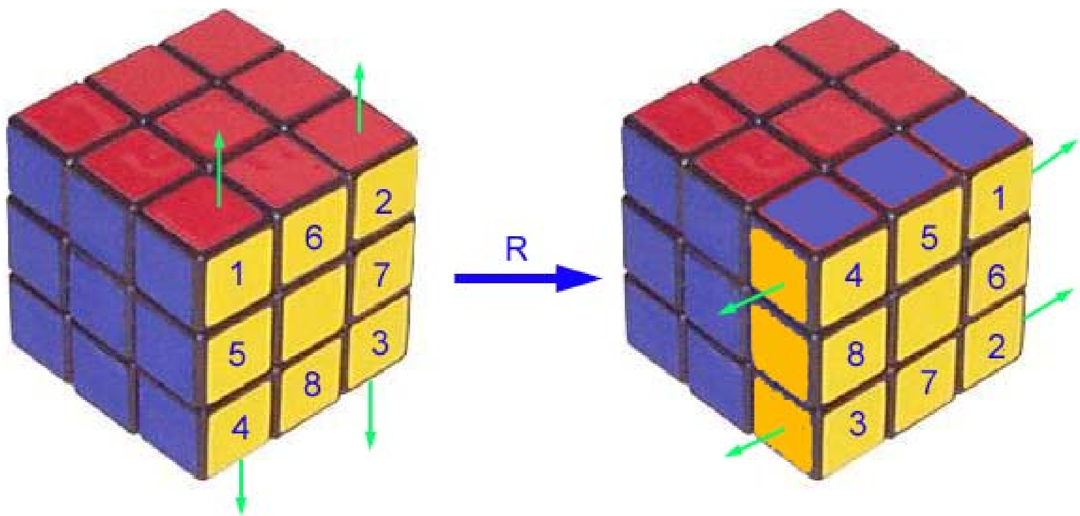
A corner cubelet can be unrotated, rotated clockwise 120 degrees, or rotated clockwise 240 degrees.



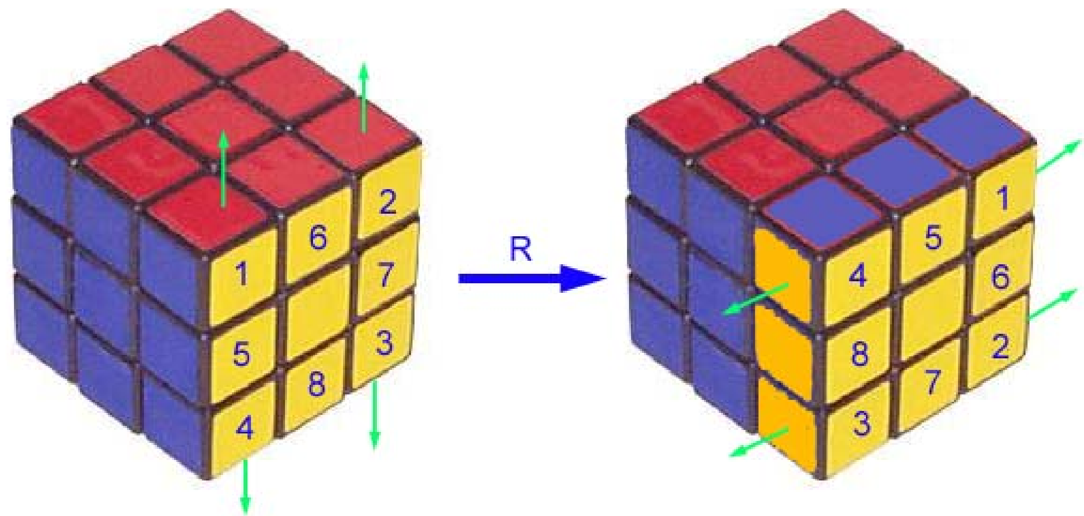
Now consider the cube below on the left. If I rotate either the top or the bottom face, no changes occur in the orientations of the corner cubelets.



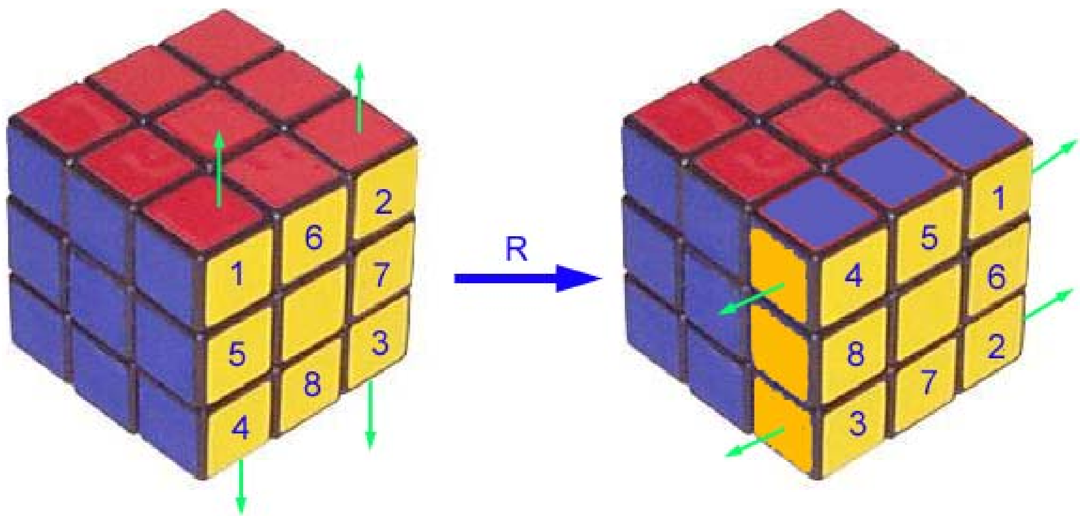
However, if I rotate the right face a quarter turn clockwise, then changes occur in my orientations.



Cubelet 1 has been rotated 120 degrees clockwise, cubelet 2 has rotated 240 degrees, cubelet 3 has rotated 120 degrees, and cubelet 4 has rotated 240 degrees.

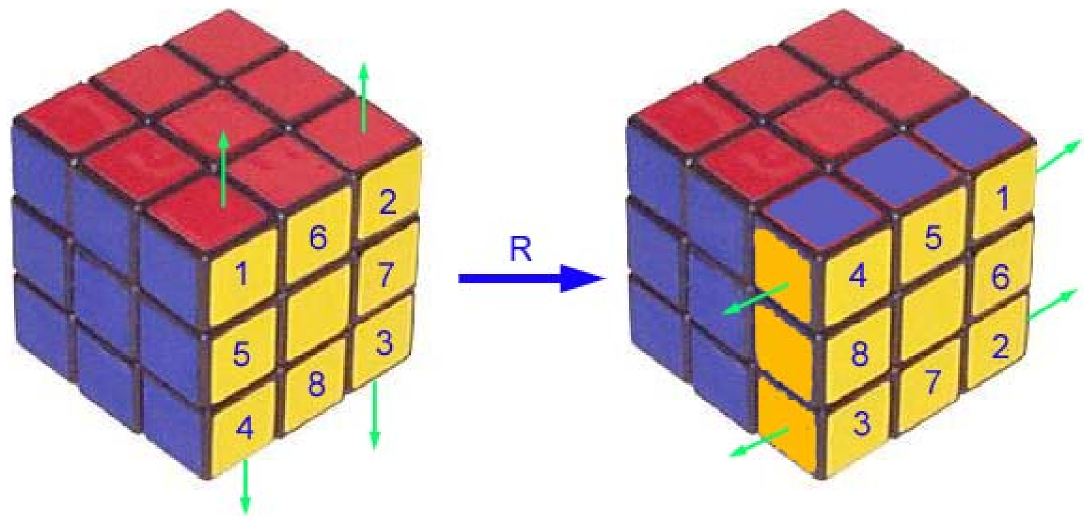


Overall, the orientation changes of the corner cubelets are a multiple of 360 degrees. We'll call this orientation 1.



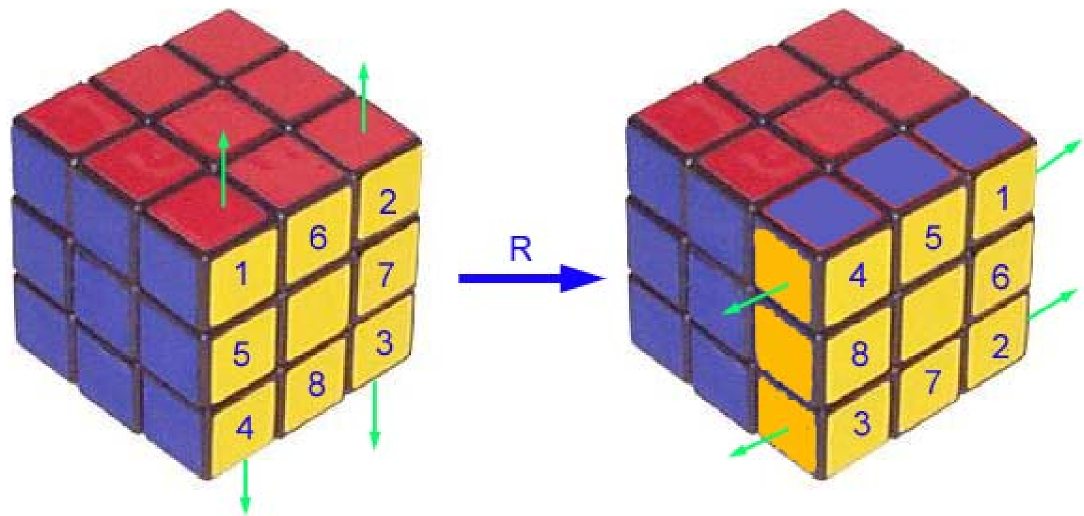
$$120° + 240° + 120° + 240° = 720° = 2 \cdot 360°$$

And anytime we rotate a face, the orientation changes of the corner cubelets will always be a multiple of 360 degrees. The collective corner cubelets will always have orientation 1.



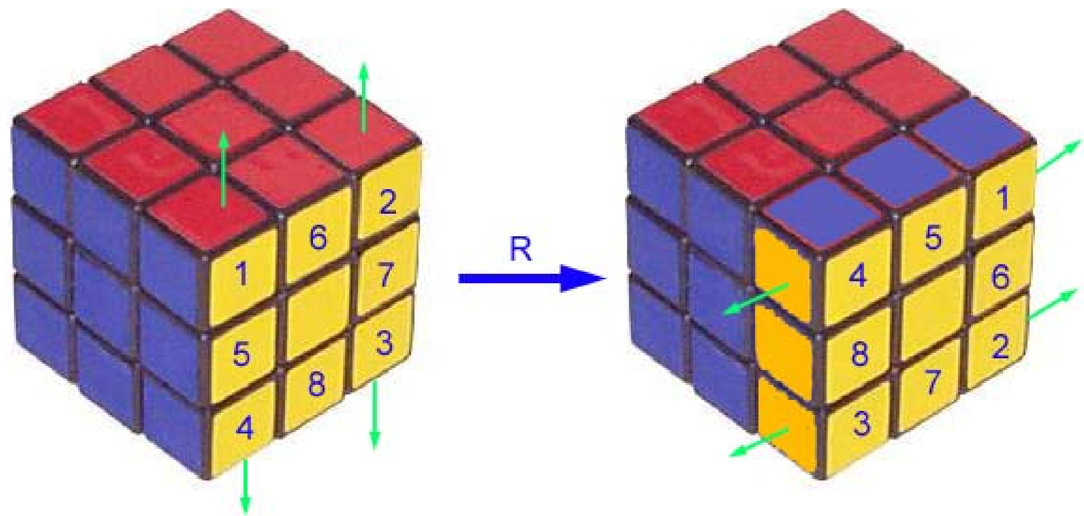
$$120° + 240° + 120° + 240° = 720° = 2 \cdot 360°$$

If the rotations of the corner cubelets added up to a multiple of 360 degrees plus 120 degrees, we would call that orientation 2.



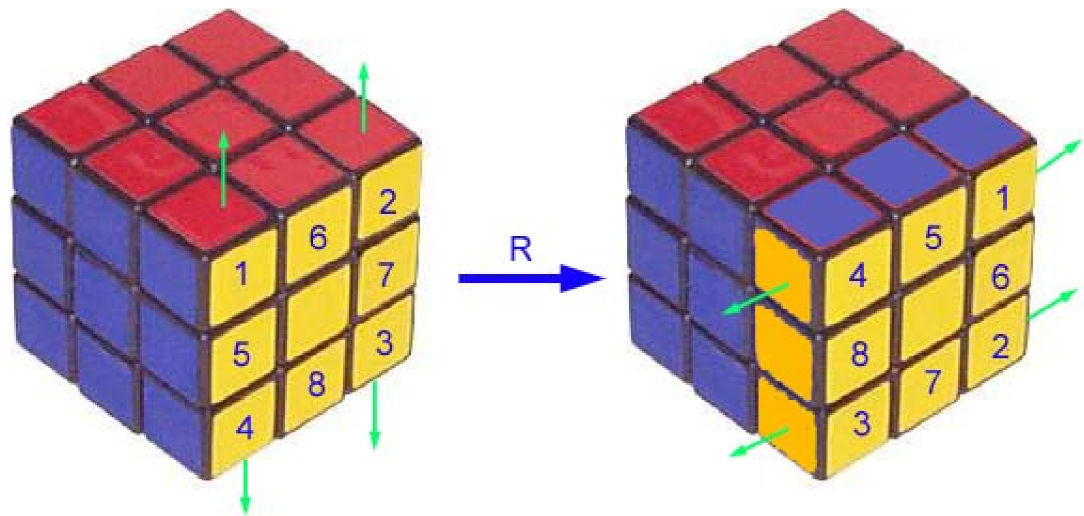
$$120° + 240° + 120° + 240° = 720° = 2 \cdot 360°$$

And if the rotations of the corner cubelets added up to a multiple of 360 degrees plus 240 degrees, we would call that orientation 3.



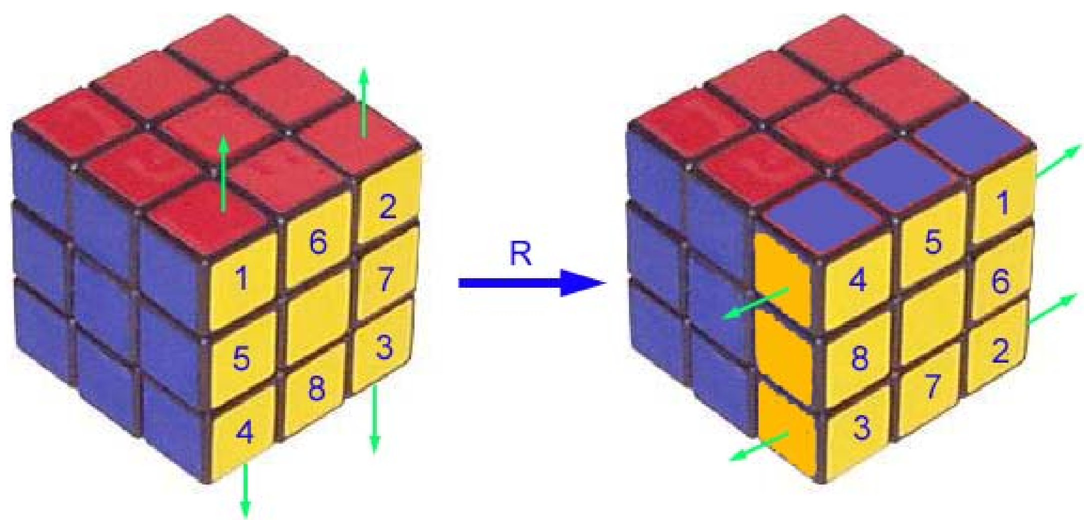
$$120° + 240° + 120° + 240° = 720° = 2 \cdot 360°$$

However, our collective corner cubelets will always have orientation 1, and thus, the correct number of possible permutations of the facelets in Rubik's cube is only a third of our previous estimate.



$$\frac{(8!)(12!)(3^8)(2^{12})}{2 \cdot 2 \cdot 3} = 43,252,003,274,489,856,000 = 2^{27}3^{14}5^{3}7^{2}11$$

This brings the number of possible permutations of the facelets of Rubik's cube to a little over 43 quintillion, and how do we know that this number still isn't too large?



$$\frac{(8!)(12!)(3^8)(2^{12})}{2 \cdot 2 \cdot 3} = 43{,}252{,}003{,}274{,}489{,}856{,}000 = 2^{27}3^{14}5^3 7^2 11$$

Because Chuck Norris has actually done all 43,252,003,274,489,856,000 permutations!

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- **Chuck Norris has also counted to infinity.**

Because Chuck Norris has actually done all 43,252,003,274,489,856,000 permutations!

- **Chuck Norris has also counted to infinity.**

- **TWICE!**

Because Chuck Norris has actually done all 43,252,003,274,489,856,000 permutations!

- Chuck Norris has also counted to infinity.

- TWICE!

- I’ve counted to infinity only once, but I counted backwards.

Because Chuck Norris has actually done all 43,252,003,274,489,856,000 permutations!

- **Chuck Norris has also counted to infinity.**
- **TWICE!**
- **I’ve counted to infinity only once, but I counted backwards.**
- **And Chuck Norris CAN divide by zero!**

THE END