ISOMORPHISMS AND HOMOMORPHISMS

We've used the term "isomorphism" before, and we pointed out that it literally means "equal shape." We've also said that two groups are isomorphic if they are essentially the same group, but with different labels for the elements. That means that there has to exist a correspondence between the two groups that is both one-to-one and onto, what we call a "bijection." Additionally, an isomorphism between two groups also means that multiplication in one group has to correspond to multiplication in the other group. So far we've avoided using functions to define isomorphisms in order to keep things a little simpler. However, the time has come to streamline our thinking by giving an explicit definition of a isomorphism purely in terms of a function from one group onto another. First, though, we will also give more specific definitions for terms like one-to-one, onto, injection, surjection, and bijection.

<u>Definition:</u> Let $f: A \to B$ be a function. Then *f* is one-to-one if and only if whenever we have $x, y \in A$ with $x \neq y$, we also have that $f(x) \neq f(y)$. Equivalently, we can say that *f* is one-to-one if $f(x) = f(y)$ always implies that $x = y$. A one-to-one function is also known as an <u>injection</u> or <u>injective function</u>.

<u>Definition:</u> Let $f: A \to B$ be a function. Then *f* is an onto function if and only if whenever $y \in B$, there exists $x \in A$ such that $f(x) = y$. An onto function is also known as a <u>surjection</u> or <u>surjective function</u>.

<u>Definition:</u> Let $f: A \to B$ be a function. If *f* is both one-to-one and onto (both injective and surjective), then *f* is also called a <u>bijection</u> or <u>bijective function</u>.

<u>Definition:</u> Let $f: A \to B$ be a bijective function from a group *A* onto a group *B*. Then *f* is also an isomorphism if for all $x, y \in A$, we have that $f(x)f(y) = f(xy)$. Note that this basically says that if $xy = z$ in *A*, then $f(x)f(y) = f(xy) = f(z)$ in *B*. In other words, multiplication in *A* corresponds to multiplication in *B*.

A concept of that is more general than that of an isomorphism is the notion of a "homomorphism." The word itself literally means "same shape," and the difference between an isomorphism and a homomorphism is that we drop the condition that our function be one-to-one. We will keep, however, the onto condition.

<u>Definition:</u> Let $f: A \to B$ be a surjective function from a group *A* onto a group *B*. Then *f* is also a homomorphism if for all $x, y \in A$, we have that $f(x)f(y) = f(xy)$. Note that this basically says that if $xy = z$ in *A*, then $f(x)f(y) = f(xy) = f(z)$ in *B*. In other words, multiplication in *A* corresponds to multiplication in *B*.

An example of a homomorphism that is not an isomorphism would be the function that takes every integer in the group of integers under addition, and assigns that integer either to the label "even" or to the label "odd" in the usual manner. Suppose we call this latter set *E* and that we define multiplication in *E* according to the following table.

+	**Even**	**Odd**
Even	Even	Odd
Odd	Odd	Even

Then *E* is a group of order 2, and our function $f:\mathbb{Z}\to E$ is a homomorphism. Hence, using additive rather than multiplicative notation, we have, for instance, that $f(2)+f(3)=even+odd=odd=f(2+3)=f(5)$. In other words, there is a correspondence between addition in $\mathbb{Z}$ and addition in *E*.

And finally, if we do have a homomorphism $f:A\to B$, then of particular concern will be the elements in *A* that get sent or mapped to the identity element in *B*. The set of such elements in *A* is called the "kernel of our homomorphism *f*."

<u>Definition:</u> Let $f:A\to B$ be a homomorphism from *A* onto *B*. Then the <u>kernel of *f*</u>, denoted by $Ker(f)$, is defined by

$Ker(f)=\{x\in A \mid f(x)=e \text{ where } e \text{ is the identity element in } B\}$.