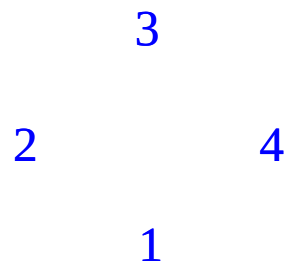
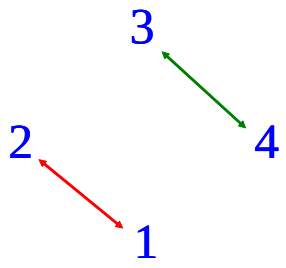
Lesson 12

INTRODUCTION TO ORBITS

A very simple concept, but one which has far reaching implications is the notion of an orbit. The concept begins with a set of objects or points that a group *G* acts on. For example, if we are looking at permutations of the set of points $\{1,2,3,4\}$, then the orbit of 1 is the subset of all numbers that 1 can be transformed into by the permutations contained in our group *G*. We define the orbits of 2, 3, and 4 similarly. Also, it should be clear that if 2 is in the orbit of 1, then 1 is in the orbit of 2. This is because if we have a permutation that transforms 1 into 2, then the inverse of that permutation will transform 2 back into 1. The bottom line is that our original set of points can be decomposed into a series of non-overlapping subsets whose union gives us back the entire set of points. Furthermore, each subset represents the mutual orbit for the points in that subset.



Here are a few examples. Consider, first, the generator diagram below. It should be clear from this diagram that we can change 1 to 2 and 2 to 1, and so 1 & 2 are in the same orbit. Likewise, we can change 3 to 4 and 4 to 3, and so that gives us a second orbit. However, there is no way we can use the indicated moves in our diagram to change 1 to 3. Thus, we have two distinct, non-intersecting orbits whose union is the set $\{1,2,3,4\}$.



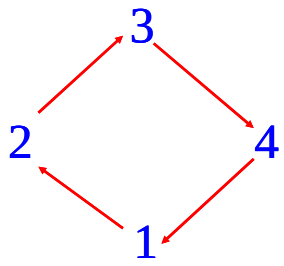
G = { (), (3,4), (1,2), (1,2)(3,4) }

$Orbit(G,1) = \{1,2\}$

$Orbit(G,3) = \{3,4\}$

$Orbits(G) = \{\{1,2\},\{3,4\}\}$

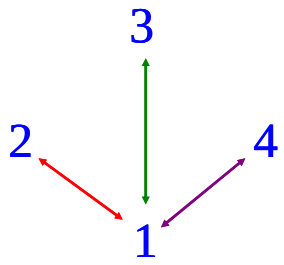
However, in our next generator diagram, we should be able to move 1 to any other number using the indicated moves. Thus, for the generated group *G* that acts upon this set there should be only one orbit.



G = { (), (1,2,3,4), (1,3)(2,4), (1,4,3,2) }

$Orbit(G,1) = \{1,2,3,4\}$

And finally, this generator diagram also results in just one orbit. Again, the resulting permutations in *G* allow us to move any number to any other number.



G = { (), (3,4), (2,3), (2,3,4), (2,4,3), (2,4), (1,2), (1,2)(3,4), (1,2,3), (1,2,3,4), (1,2,4,3), (1,2,4), (1,3,2), (1,3,4,2), (1,3), (1,3,4), (1,3)(2,4), (1,3,2,4), (1,4,3,2), (1,4,2), (1,4,3), (1,4), (1,4,2,3), (1,4)(2,3) }

$Orbit(G,1) = \{1,2,3,4\}$