Lesson 10

INTRODUCTION TO CONJUGATES

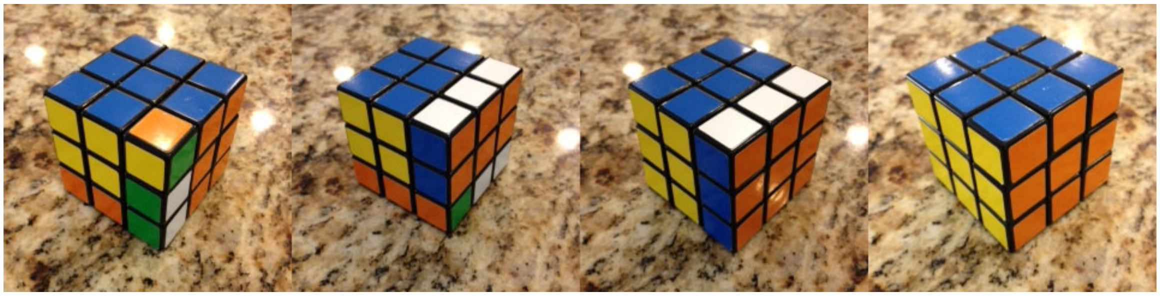
If a and b are both elements of a group G and if a and b commute with one another, then we have that $ab = ba$. Another way to write this is as $b^{-1}ab = a$. Of course, what often happens in nonabelian groups is that if we randomly pick two elements a and b, then it will probably be the case that $b^{-1}ab \neq a$. Either way, we call $b^{-1}ab$ the conjugate of a by b, and this is often written in group theory texts as $a^b = b^{-1}ab$. We should note, though, that many textbooks define the conjugate of a by b as bab^{-1} rather than $b^{-1}ab$. Ultimately, it doesn't make that much difference which definition you use, but since the computer program GAP (Groups, Algorithms, and Programming) that we'll talk about later defines it as $a^b = b^{-1}ab$, that's the definition we'll use throughout.

Here is a quick example from the symmetric group S_3, the group of all permutations of three objects. If $a = (1,2)$ and $b = (1,2,3)$, then the conjugate of a by b is $b^{-1}ab = (3,2,1)(1,2)(1,2,3) = (1)(2,3) = (2,3)$. Hence, $(2,3)$ is a conjugate of $(1,2)$. Likewise, $(1,2)$ is a conjugate of $(2,3)$ by $b^{-1} = (3,2,1)$
since $(2,3)^{b^{-1}} = b \cdot (2,3) \cdot b^{-1} = (1,2,3)(2,3)(3,2,1) = (1,2)$.

There are a couple of reasons why conjugates are important, and we'll begin with an application involving Rubik's cube. There is a group that naturally acts on the cubelets comprising Rubik's cube, and the motions that generate it are defined as follows:

R = rotate the right face 90° clockwise
L = rotate the left face 90° clockwise
U = rotate the up face 90° clockwise
D = rotate the down face 90° clockwise
F = rotate the front face 90° clockwise
B = rotate the back face 90° clockwise

In a similar manner we can define R^{-1} to mean "rotate the right face 90° counterclockwise." Thus, suppose on our Rubik's cube that we have a cubelet on the down-left face on our lower left that we want to move to the right-up face on our upper right as we look at the cube. Then we can accomplish this move by performing $R^{-1}DR$. Basically, R^{-1}, rotating the right face counterclockwise, takes the cubelet we want to change to a "workspace" at the bottom of the cube. We then perform D, rotating the down face 90° clockwise, to move our new cubelet into position, and we do R, rotating the right face 90° clockwise, to move the cubelet to the top of the cube on the right side. Here is a picture that illustrates what I'm talking about.



$R^{-1}DR$

So, the above gives us a practical application for conjugates. However, there are also theoretical reasons why conjugates are important, and the main one is that they tend to preserve structure within the group. In other words, we can take conjugates not only of individual elements, but also of subgroups or of all the elements in our group. For example, consider the following group along with a subgroup *H*.

$$S_3 = \left\{ \begin{array}{c} (\ ) \\ (1,2,3) \\ (1,3,2) \\ (1,2) \\ (2,3) \\ (1,3) \end{array} \right\} \quad \text{and} \quad H = \left\{ \begin{array}{c} (\ ) \\ (1,2,3) \\ (1,3,2) \end{array} \right\}$$

Also, let $b = (1,2)$, and notice that $b = b^{-1}$. Hence,

$$H^b = b^{-1}Hb = (1,2)\left\{ \begin{array}{c} (\ ) \\ (1,2,3) \\ (1,3,2) \end{array} \right\}(1,2) = \left\{ \begin{array}{c} (1,2)(\ )(1,2) \\ (1,2)(1,2,3)(1,2) \\ (1,2)(1,3,2)(1,2) \end{array} \right\} = \left\{ \begin{array}{c} (\ ) \\ (1,3,1) \\ (1,2,3) \end{array} \right\}$$

Notice that in this case, we got back our same subgroup *H* even though some of the elements inside were changed from one permutation to another. When this happens, no matter what element *b* we pick from our group, then we say that the subgroup *H* is a <u>self-conjugate</u> or <u>normal subgroup</u> of our main group. Now let's look at an example of a group *G* that has a subgroup *H* that is not normal (not self-conjugate).

$$G = \left\{ \begin{matrix} (\,) \\ (1,2,3) \\ (1,3,2) \\ (2,3,4) \\ (2,4,3) \\ (1,2,4) \\ (1,4,2) \\ (1,3,4) \\ (1,4,3) \\ (1,2)(3,4) \\ (1,3)(2,4) \\ (1,4)(2,3) \end{matrix} \right\} \quad \text{and} \quad H = \left\{ \begin{matrix} (\,) \\ (1,2,3) \\ (1,3,2) \end{matrix} \right\}$$

Let's let $b = (1,2)(3,4)$. Then once again $b^{-1} = b = (1,2)(3,4)$. It will not always be the case for every element *b* in every group that $b^{-1} = b$, but to keep things simple, that's how it is in the examples we've picked. Furthermore,

$$H^b = b^{-1}Hb = (1,2)(3,4)\left\{ \begin{matrix} (\,) \\ (1,2,3) \\ (1,3,2) \end{matrix} \right\}(1,2)(3,4) = \left\{ \begin{matrix} (1,2)(3,4)(\,)(1,2)(3,4) \\ (1,2)(3,4)(1,2,3)(1,2)(3,4) \\ (1,2)(3,4)(1,3,2)(1,2)(3,4) \end{matrix} \right\} = \left\{ \begin{matrix} (\,) \\ (1,4,2) \\ (1,2,4) \end{matrix} \right\} = H_2$$

In this case, the subgroup *H* is not self-conjugate, but we do see that that the conjugate of our subgroup *H* is simply another subgroup H_2. Furthermore, this new subgroup H_2 has the same number of elements as *H*, and the basic structure of the cycles is preserved under the conjugation by $b = (1,2)(3,4)$. This is what we mean when we say that conjugation preserves the basic structure of the group. Subgroups are taken by conjugation to subgroups that have the same order and the same cycle structure.