Integrating Factors

$$\frac{dy}{dx} + P(x)y = Q(x)$$

If we look at the left-hand side of the equation below, it almost looks like what we might expect when we differentiate using the product rule.

$$\frac{dy}{dx} + P(x)y = Q(x)$$

$$f(x) \cdot \frac{dg}{dx} + g(x) \cdot \frac{df}{dx}$$ Product Rule

Our goal now is to find a function u(x) that we can multiply by in order to turn the left-hand side into the derivative of a product.

$$u(x)\cdot\frac{dy}{dx}+u(x)\cdot P(x)y=\frac{d(u(x)\cdot y)}{dx}$$

$u(x)$ is called an integrating factor.

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$$u(x)=Ce^{\int P(x)dx}$$

And now it's fairly simple to find a formula for solving our original differential equation.

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And it's just that simple!

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Check:

$$\frac{dy}{dx} + 2y = \frac{d(2 + Ce^{-2x})}{dx} + 2(2 + Ce^{-2x})$$

$$= -Ce^{-2x} + 4 + 2Ce^{-2x}$$

$$= 4$$

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Check:

$$\frac{dy}{dx} + \tan(x)y$$

$$= \frac{d(1 + C\cos x)}{dx} + \tan(x)\left(1 + C\cos x\right)$$

$$= -C\sin x + \tan x + C\cos x \tan x$$

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$$= \tan x$$