infinity

from L. *infinitatem* (nom. *infinitas*) "boundlessness," from *infinitus* boundless, unlimited"

Hindu

This universe existed in the shape of darkness, unperceived, destitute of distinctive marks, unattainable by reasoning, unknowable, wholly immersed, as it were, in deep sleep. Then the Divine Self-existent, himself indiscernible but making all this, the great elements and the rest, discernible, appeared with irresistible power, dispelling the darkness. He who can be perceived by the internal organ alone, who is subtle, indiscernible, and eternal, who contains all created beings and is inconceivable, shone forth of his own will. He, desiring to produce beings of many kinds from his own body, first with a thought created the waters, and placed his seed in them. That seed became a golden egg, in brilliancy equal to the sun; in that egg he himself was born as Brahma, the progenitor of the whole world....

Japanese

Before the heavens and the earth came into existence, all was a chaos, unimaginably limitless and without definite shape or form. Eon followed eon: then, lo! out of this boundless, shapeless mass something light and transparent rose up and formed the heaven. This was the Plain of High Heaven, in which materialized a deity called Ame-no-Minaka-Nushi-no-Mikoto (the Deity-of-the-August-Center-of-Heaven). Next the heavens gave birth to a deity named Takami-Musubi-no-Mikoto (the High-August-Producing-Wondrous-Deity), followed by a third called Kammi-Musubi-no-Mikoto (the Divine-Producing-Wondrous-Deity). These three divine beings are called the Three Creating Deities.

Tao te Ching

There is something formless and perfect,
Existing before the birth of Heaven and Earth.
How still it is! How quiet!
Abiding alone and unchanging.
It pervades everywhere without fail.
Well may it be called mother of the world.
I do not know its name,
But label it Tao.

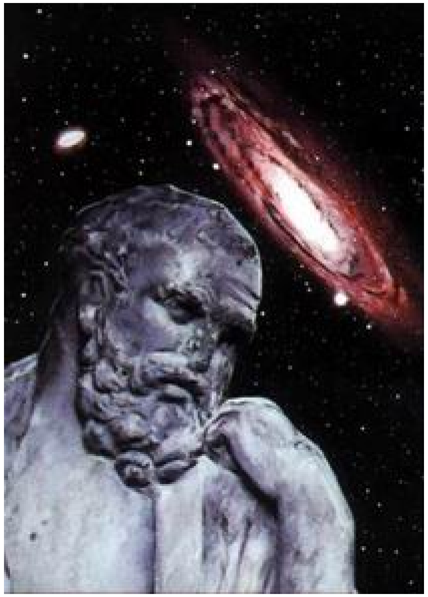
Ovid’s Metamorphoses:

“Before the ocean was, or earth, or heaven,
Nature was all alike, a shapelessness,
Chaos, so-called, all rude and lumpy matter,
Nothing but bulk, inert, in whose confusion
Discordant atoms warred, . . . Till God, or kindlier
Nature,settled all argument, and separated
Heaven from earth, water from land, our air
From the high stratosphere, a liberation
So things evolved, and out of blind confusion
Found each its place, bound in eternal order.”

Anaximander of Miletus

610 BCE – 546 BCE

- **All things originate in *apeiron* (boundlessness).**

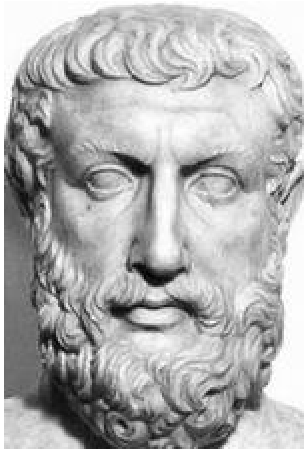


Anaxagoras

500 BCE – 428 BCE

- All things originate in *apeiron*.
- Everything contains everything else.
- Mind (Nous) separates things from *apeiron*.

Parmenides of Elea
515 BCE



- **Reality is an unchanging unity.**
- **Change is an illusion.**
- **What is is, what ain’t ain’t.**

Zeno of Elea
490 BCE

→**Student of Parmenides**

→**Famous for his paradoxes**

- Achilles and the Tortoise
- The Arrow
- The Stadium

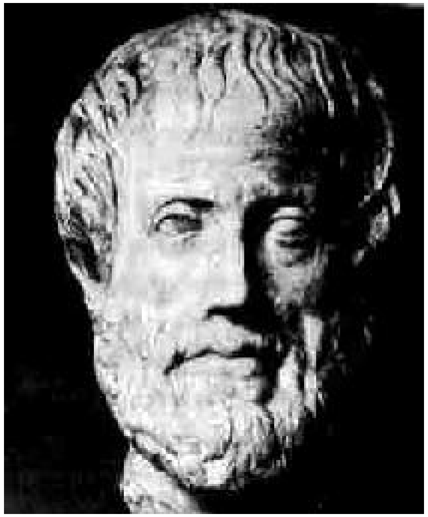
Zeno Type Paradoxes

•If you make a book with an infinite number of pages such that the first page is ½ inch thick, the second ¼, and so on, what will you see when you turn the book over?

•If you turn a light on after ½ a minute, off after ¼, and so on, will it be on or off after 1 minute?

•Suppose space & time are finitely divisible. Then there is a maximum velocity. Suppose A & B are traveling toward each other at maximum velocity. However, if A perceives itself as being stationary, B must be traveling at twice the maximum velocity.

Aristotle
384 BCE – 322 BCE



Aristotle made a distinction between the potential infinite and the actual infinite.

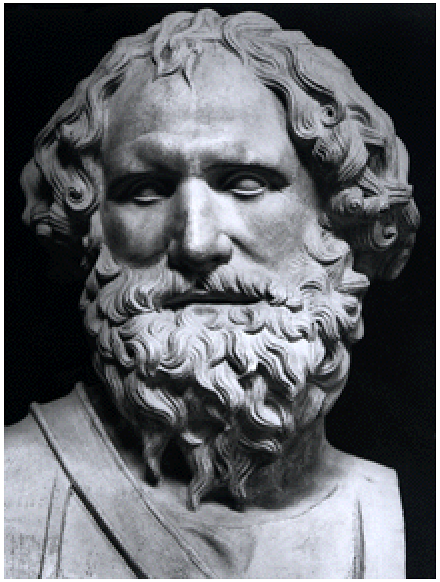
"When we speak of the potential existence of a statue we mean that there will be an actual statue. It is not so with the infinite. There will not be an actual infinite."

–Aritstotle's *Physics*

Euclid
circa 300 BCE

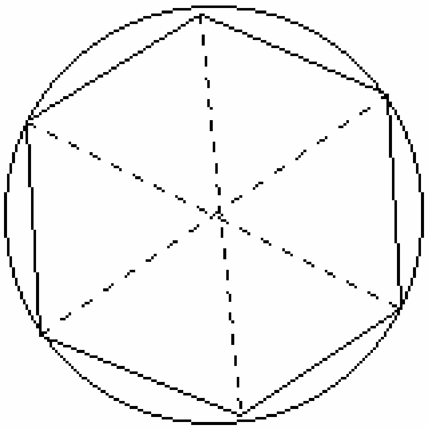
1. **A straight line segment can be drawn joining any two points.**
2. **Any straight line segment can be extended indefinitely in a straight line.**
3. **Given any straight line segment, a circle can be drawn having the segment as radius and one endpoint as center.**
4. **All right angles are congruent.**
5. **Given any straight line and a point not on it, there "exists one and only one straight line which passes" through that point and never intersects the first line, no matter how far they are extended.**

Archimedes
287 BCE – 212 BCE



Archimedes used the method of exhaustion of Eudoxus (408 BCE) to find the area of a circle.

$$A = n\left(\frac{bh}{2}\right) = \frac{(nb)h}{2} \approx \frac{(2\pi r)r}{2} = \pi r^2$$



Galileo Galilei
1564 CE – 1642 CE

Galileo's Paradox:

There are just as many perfect squares as there are counting numbers.

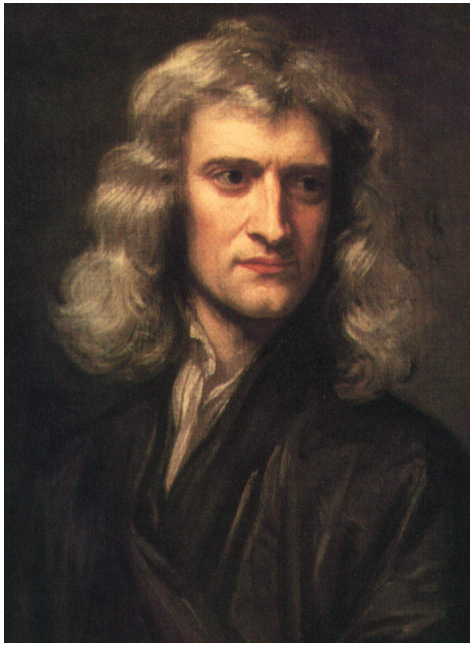
$$\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 \ldots & n \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 1 & 4 & 9 & 16 & 25 & n^2 \end{array}$$

Carl Friedrich Gauss
1777 CE – 1855 CE

I protest against the use of infinite magnitude as something completed, which is never permissible in mathematics. Infinity is merely a way of speaking, the true meaning being a limit which certain ratios approach indefinitely close, while others are permitted to increase without restriction.

Isaac Newton

1642 CE – 1727 CE

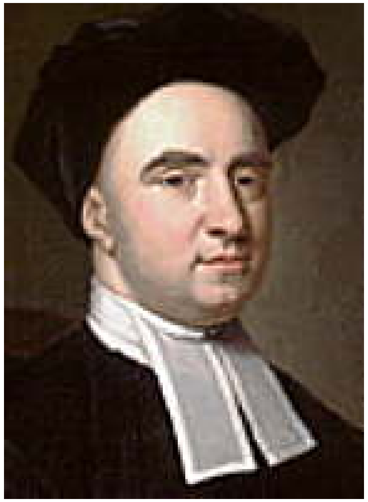


Gottfried Wilhelm Leibniz

1646 CE – 1716 CE

An **infinitesimal** is a number that is smaller in absolute value than any positive real number.

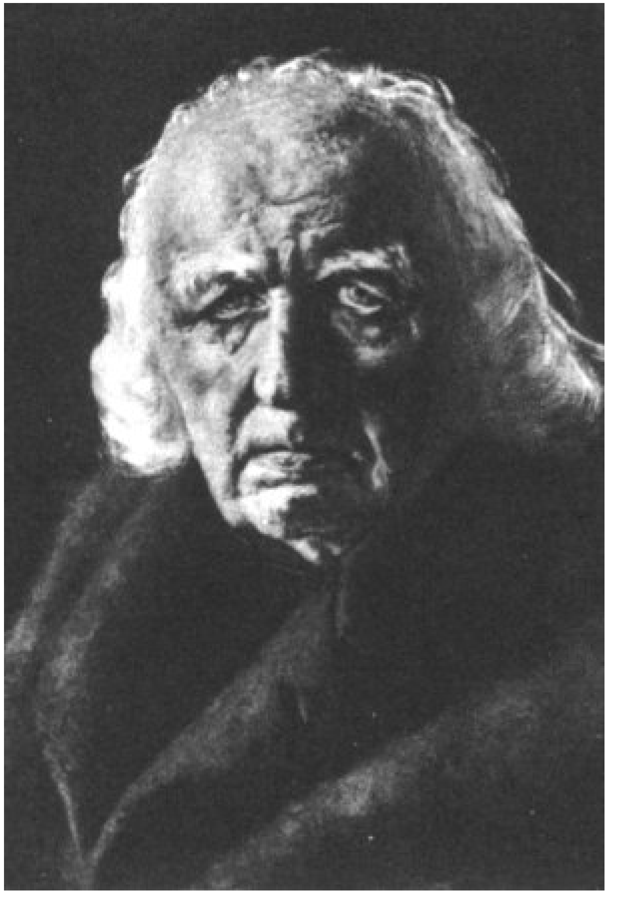
Bishop George Berkeley
1685 CE – 1753 CE



THE
ANALYST;
OR, A
DISCOURSE
Addressed to an
Infidel MATHEMATICIAN.
WHEREIN
It is examined whether the Object, Principles, and Inferences of the modern Analysis are more distinctly conceived, or more evidently deduced, than Religious Mysteries and Points of Faith.
By George Berkeley
(1734)

Karl Weierstrass
1815 CE – 1897 CE

The Father of Modern Analysis



"Until Weierstrass, mathematicians had no limit."
-C. Benton

$$\lim_{x\to a} f(x) = L \Leftrightarrow \forall\varepsilon > 0, \exists\delta > 0 \ni 0 < |x-a| < \delta \Rightarrow |f(x) - L| < \varepsilon$$

Georg Cantor
1845 CE – 1918 CE

Creator of Set Theory

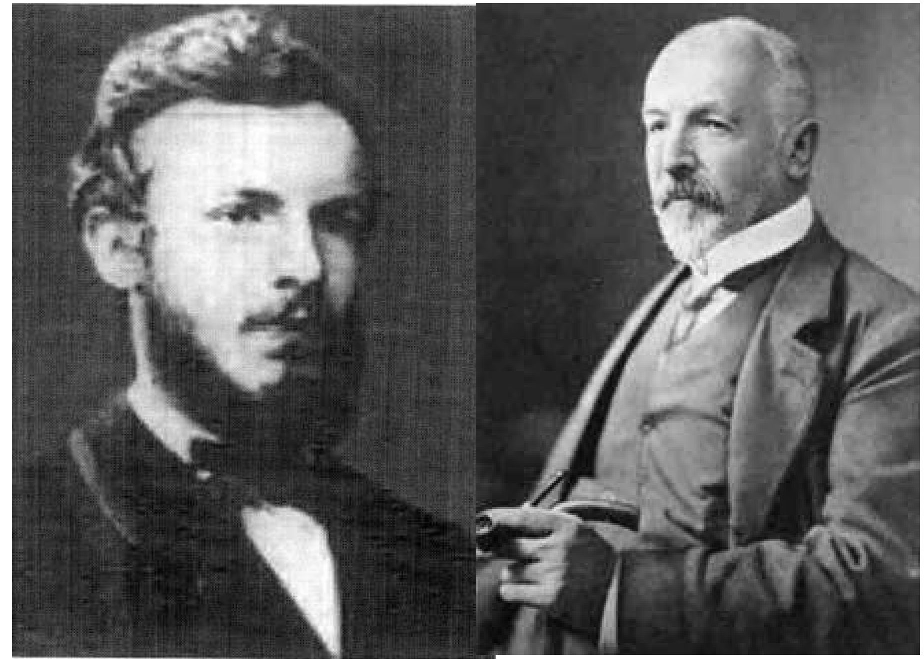
“A set is a many which can be thought of as a one.”

Georg Cantor
1845 CE – 1918 CE

Creator of Set Theory

"The fear of infinity is a form of myopia that destroys the possibility of seeing the actual infinite, even though it in its highest form has created and sustains us, and in its secondary transfinite forms occurs all around us and even inhabits our minds."

The set of counting numbers cannot be put into one-to-one correspondence with the real numbers in the interval (0,1).



$$1 \rightarrow 0.1451939\ldots$$

$$2 \rightarrow 0.5876624\ldots$$

$$3 \rightarrow 0.9942146\ldots$$

$$4 \rightarrow 0.3621722\ldots$$

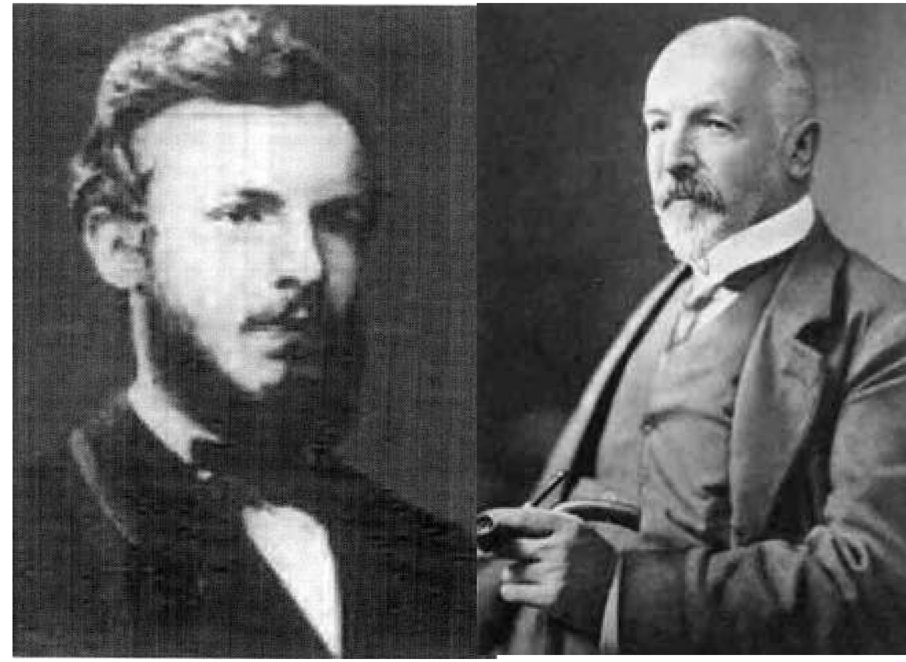
$$n \overset{?}{\rightarrow} 0.2112\ldots$$

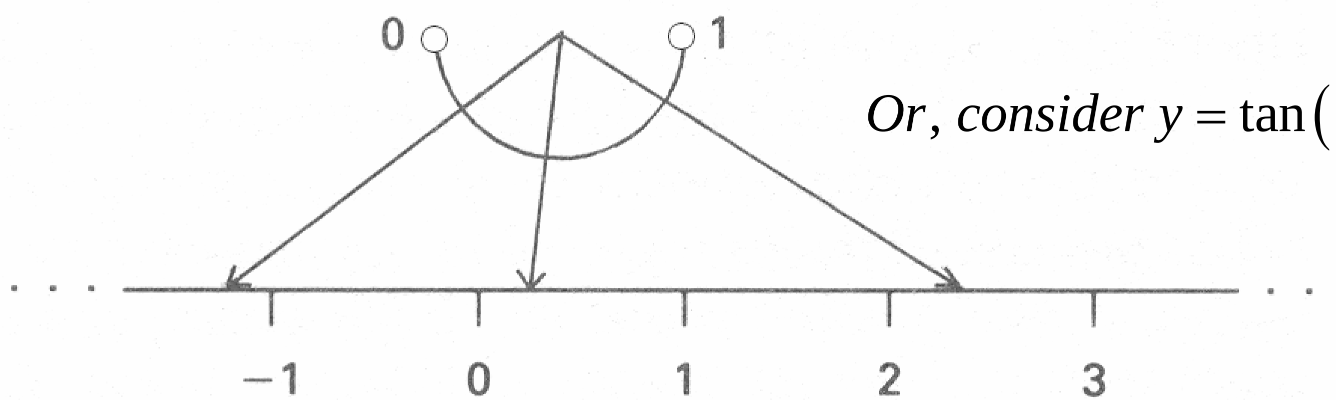
There are just as many numbers in the interval (0,1) as there are in the open square (0,1) x (0,1).

“I see it, but I don’t believe it.” -Cantor

$$0.{\color{red}2}{\color{green}8}{\color{red}7}{\color{green}6}{\color{red}5}{\color{green}4}{\color{red}3}{\color{green}9}{\color{red}1}{\color{green}2}{\color{red}2}{\color{green}3} \to \left(0.{\color{red}275312}, 0.{\color{green}864923}\right)$$

There are just as many numbers in the interval (0,1) as there are real numbers.





$Or,\ consider\ y = \tan\left(\pi x - \pi/2\right)$

The set of all subsets of the counting numbers cannot be put into one-to-one correspondence with the counting numbers.

	1	2	3	.	.	.
1	***yes***	***no***	***yes***			
2	***no***	***no***	***yes***			
3	***no***	***no***	***no***			
.						
.						
.						

$$n \overset{?}{\rightarrow} no, yes, yes, \ldots$$

The set of all subsets of a set A cannot be put into one-to-one correspondence with the elements of A.

Let A be a set and let B be the set of all subsets of A. Since the result is obvious when A is empty, assume A is non-empty. Now assume that f is a bijection from A to B, and let T be the set of all elements x in A such that x is not an element of f(x). Since f is a bijection, there exists an element t in A such that f(t) = T. Now ponder the question is t an element of T? Bummer. Therefore, no bijection exists from A to B.

Not only do infinite sets exist, but there are an infinite number of infinities of infinitely many different sizes

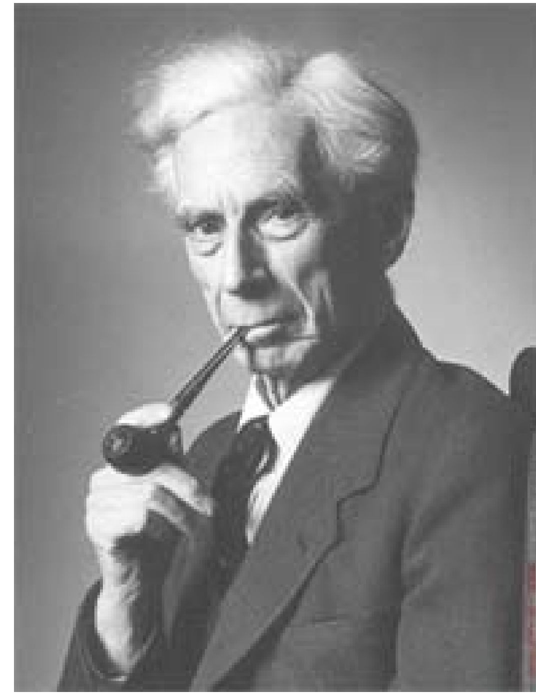
$$\left|\mathbb{N}\right| = \aleph_0 < \aleph_1 < \aleph_2 < \aleph_3 < \ldots$$

Leopold Kronecker
1823 CE – 1891 CE

$$\frac{d^3 s}{dt^3} \rightarrow$$

“God made the integers; all else is the work of man.”

Bertrand Russell
1872 – 1970



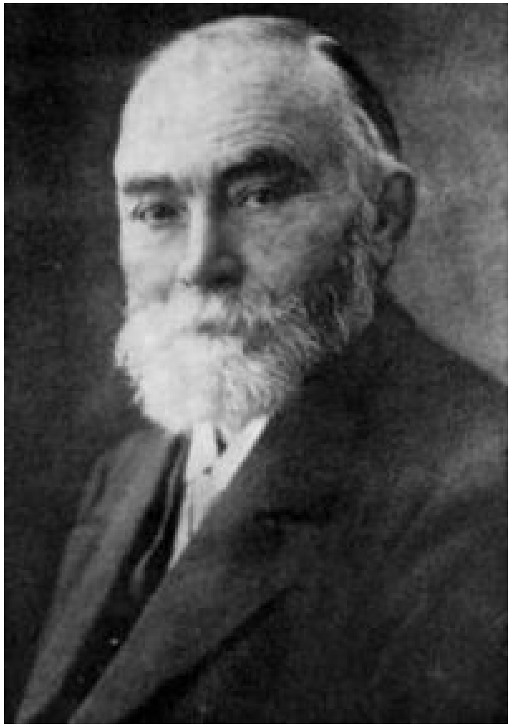
<u>Paradox Lost</u>

<u>Cantor's Paradox:</u> Let A be the set of all sets and let B be the set of all subsets of A. Then $|A| < |B|$. But since B is a subset of A, $|B| \leq |A|$.

<u>Russell's Paradox:</u> Let M be the set of all sets that do not contain themselves as elements. Then is M an element of M?

If $M = \left\{ A \middle| A \notin A \right\}$, then is $M \in M$?

Gottlob Frege
1848 CE – 1925 CE



"A scientist can hardly meet with anything more undesirable than to have the foundation give way just as the work is finished. I was put in this position by a letter from Mr. Bertrand Russell when the work was nearly through the press." -Gottlob Frege

Ernst Zermelo
1871 CE – 1953 CE

- Published an axiom system for set theory in 1908
- Used informal language
- The system was improved by Fraenkel in 1922

Zermelo-Fraenkel Axioms

Axiom of Extensionality: If X and Y have the same elements, then X = Y.

Axiom of the Unordered Pair: For any *a* and *b* there exists a set {*a*,*b*} that contains exactly *a* and *b*.

Axiom of Subsets: If P is a property (with parameter *p*) then for any X and *p* there exists a set Y = {*u* in X such that P(*u*,*p*)}.

Axiom of the Sum Set: for any X there exists a set Y equal to the union of all elements of X.

Axiom of the Power Set: For any X there exists a set Y, the set of all subsets of X.

Axiom of Infinity: There exists an infinite set.

Axiom of Replacement: If F is a function, then for X there exists a set Y = {F(*x*) such that *x* is an element of X}.

Axiom of Foundation: Every nonempty set A contains an element B which is disjoint from A.

The Continuum Question

$$|\mathbb{N}| = \aleph_0 < \aleph_1 < \aleph_2 < \aleph_3 < \ldots$$

$$|\mathbb{R}| \overset{?}{=} \aleph_1$$

The Axiom of Choice

Every family of nonempty sets has a choice function.

Equivalences to The Axiom of Choice

Well-ordering theorem: Every set can be well-ordered.

Trichotomy: If two sets are given, then either they have the same cardinality, or one has a smaller cardinality than the other.

The Cartesian product of any nonempty family of nonempty sets is nonempty.

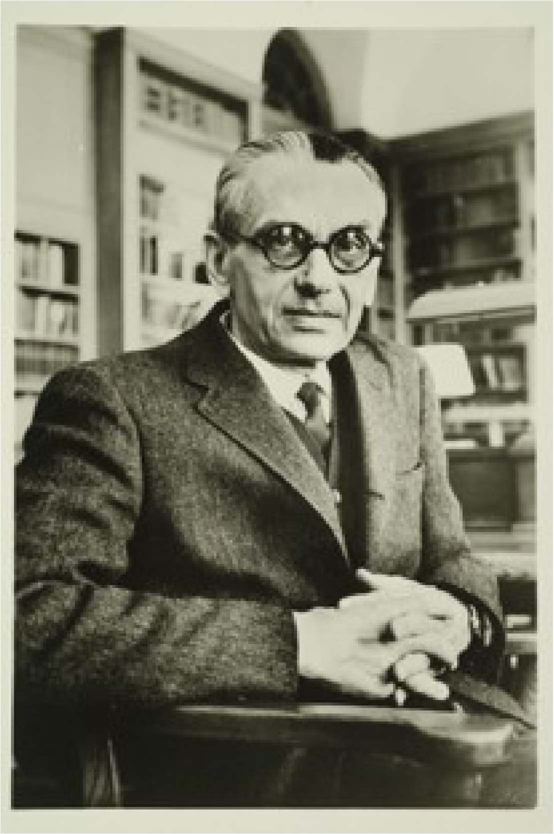
Zorn's lemma: Every non-empty partially ordered set in which every chain (i.e. totally ordered subset) has an upper bound contains at least one maximal element.

Hausdorff maximal principle: In any partially ordered set, every totally ordered subset is contained in a maximal totally ordered subset.

Every vector space has a basis.

Tychonoff's theorem stating that every product of compact topological spaces is compact.

Kurt Godel
1906 – 1978



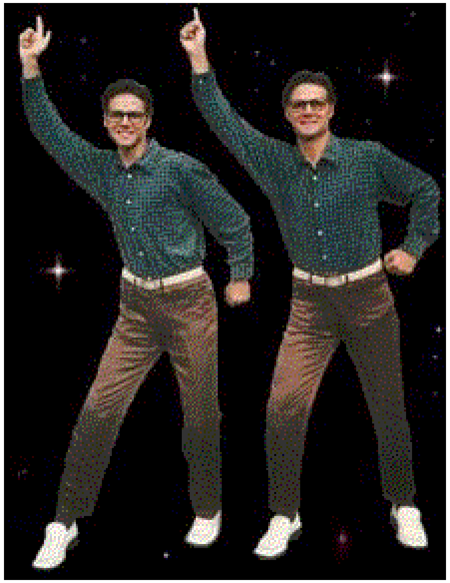
- The Continuum Hypothesis cannot be disproved in ZF.
- The Axiom of Choice cannot be disproved in ZF.
- Any consistent system rich enough to describe the arithmetic of the natural numbers will contain true statements that cannot be proven.

Paul Cohen
b. 1934 - 2007

You can't prove the Continuum Hypothesis and the Axiom of Choice in Zermelo-Fraenkel.

The Continuum Hypothesis and the Axiom of Choice are undecideable in ZF!

Paul Cohen
b. 1934 - 2007



Paul Cohen's twin sons, Steve and Eric, appeared regularly as the dancing twins on "Ally McBeal."

Henri Poincare
1854 - 1912

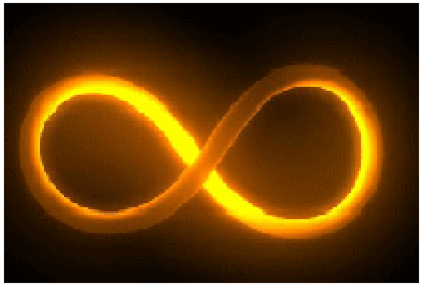
$$\frac{d^3s}{dt^3} \rightarrow$$

“Set theory is a disease, from which I hope future generations will recover."

"Point set topology is a disease from which the human race will soon recover.“

"Mathematicians do not study objects, but relations between objects.”

David Hilbert
1862 - 1943



"No one shall expel us from the paradise that Cantor has created for us."

"Mathematics is a game played according to certain simple rules with meaningless marks on paper."

"A mathematical theory is not to be considered complete until you have made it so clear that you can explain it to the first man whom you meet on the street."

Hilbert’s Hotel

The Aleph-Null Song

Aleph-null bottles of beer on the wall,

Aleph-null bottles of beer,

Take one down and pass it around,

Aleph-null bottles of beer on the wall!