HOW TO FACTOR $ax^2 + bx + c$

I now want to talk a bit about how to factor $ax^2 + bx + c$ where all the coefficients *a*, *b*, and *c* are integers. The method that most people are taught these days in high school (assuming you go to a high school where the basics of algebra are still considered important) is the trial-and-error method. In this method, you start with a trinomial such as,

$$3x^2 + 5x + 2$$

Next you reason that if it factors, then the factorization will look something like,

$$(\_\ x + \_)(\_\ x + \_)$$

From here it is easy to see that the numbers that go into the red blanks (the coefficients of *x*) must multiply to 3, and the numbers that go into the blue blanks (the constant terms) must multiply to 2. Thus, you proceed by finding all pairs of integers that multiply to 3 and all pairs that multiply to 2, and then by trial-and-error, you begin plugging those numbers into the blanks until you find a combination that gives you the right middle term. For example, $(3x+2)(x+1)$ yields the correct middle term of 5*x*, but $(3x+1)(x+2)$ results in a middle term of 7*x* which is not what we are looking for.

The advantage of the trial and error method is that it is conceptually easy to understand, but the downside is that if our constant term or our coefficient for x^2 have a lot of factors, then it can be very difficult and time consuming to find just the right one that works. Thus, I want to demonstrate and alternative method that is easier to implement and that takes all the guess work out of factoring, and then I'll show you why the method works. First, though, recall how we went about factoring a trinomial like the following where the coefficient of x^2 is 1.

$$x^2 + 5x + 6$$

Our method in this case is to find integers *u* and *v* such that $u + v = 5$ and $uv = 6$. Clearly, $u = 2$ and $v = 3$ do the trick, and one we've found *u* and *v*, we can immediately complete our factorization as follows.

$$x^2 + 5x + 6 = (x+u)(x+v) = (x+2)(x+3)$$

For a trinomial of the form $ax^2 + bx + c$, we are now going to follow similar steps. In particular:

1. Find integers *u* and *v* such that $u + v = b$ and $uv = ac$.
2. Write $ax^2 + bx + c$ as $ax^2 + ux + vx + c = (ax^2 + ux) + (vx + c)$.

3. Factor by grouping.

Here is an illustration using our original trinomial,

$$3x^2 + 5x + 2$$

We now seek integers *u* and *v* such that $u + v = 5$ and $uv = 3 \cdot 2 = 6$, and once again, $u = 2$ and $v = 3$ fit the bill. Thus, we rewrite our polynomial replacing $5x$ with $2x + 3x$, and then we factor by grouping.

$$3x^2 + 5x + 2$$

$$3x^2 + 2x + 3x + 2$$

$$(3x^2 + 2x) + (3x + 2)$$

$$x \cdot (3x + 2) + 1 \cdot (3x + 2)$$

$$(x + 1)(3x + 2)$$

And we're done! Notice in the above example all our coefficients were positive integers. If some of our coefficients are, on the other hand, negative, then the method still works, but you may have to factor out a negative sign at some point in order to complete the factoring by grouping. For instance, consider the following example.

$$3x^2 - 5x + 2$$

$$3x^2 - 2x - 3x + 2$$

$$(3x^2 - 2x) + (-3x + 2)$$

$$x \cdot (3x - 2) - 1 \cdot (3x - 2)$$

$$(x - 1)(3x - 2)$$

And now we want to understand why this method works, and to keep things simple, I will assume that all the coefficients are positive integers. And as we saw in the last example, if we do have some negative coefficients, then the only adjustment that might have to be made to the method is to factor out a negative rather than a positive common factor when doing the factoring by grouping. Hence, let's begin with $ax^2 + bx + c$ where all the coefficients are assumed to be positive integers and the greatest common factor of *a, b,* and *c* is 1. The next step is to find integers *u* and *v* such that,

$$u + v = b \text{ and } uv = ac$$

We can usually find such a pair by trial-and-error, but by treating this as a system of two equations in two unknowns, *u* and *v*, we can easily find a solution by using the quadratic formula. For example,

$$u+v=b \text{ and } uv=ac$$

$$\Rightarrow v=b-u \text{ and } uv=u(b-u)=ub-u^2=ac$$

$$\Rightarrow 0=u^2-bu+ac$$

$$\Rightarrow u=\frac{b\pm\sqrt{b^2-4ac}}{2}$$

This last formula tells us several things. First, b^2-4ac must be a perfect square since, otherwise, *u* won't be an integer. And second, we get two solutions to our equation, and if we let one of them be $u=\frac{b+\sqrt{b^2-4ac}}{2}$, then the other solution can be denoted by $v=\frac{b-\sqrt{b^2-4ac}}{2}$. However, it is quicker to get *v* by just using the equation above that $v=b-u$. The third thing we notice is that if our system of equations, $u+v=b$ and $uv=ac$, has a solution, then there is only one pair of integers that we can represent as *u* and *v*. It doesn't really matter, though, which number we designate as *u* and which as *v*, but when considered as a pair, there is only one solution to the system. This will be important to us!

Next, we want to show that if a pair of integers, *u* and *v*, exist such that $u+v=b$ and $uv=ac$, then the trinomial is definitely factorable. To see how, just follow the following chain of logic.

$$\begin{aligned}
&ax^2+bx+c\\
&=ax^2+ux+vx+c\\
&=\frac{uv}{c}x^2+ux+vx+c\\
&=\frac{1}{c}(uv\cdot x^2+uc\cdot x+vc\cdot x+c^2)\\
&=\frac{1}{c}\left[(uv\cdot x^2+uc\cdot x)+(vc\cdot x+c^2)\right]\\
&=\frac{1}{c}\left[ux(vx+c)+c(vx+c)\right]\\
&=\frac{1}{c}\left[(ux+c)(vx+c)\right]
\end{aligned}$$

At this point, notice that since the coefficients of our original trinomial are integers, the $\frac{1}{c}$ must at some point cancel out. Hence, we must be able to write c as $c = m_1 m_2$ where m_1 divides u and m_2 divides v. In other words, there exist m_1 and m_2 such that $c = m_1 m_2$, $u = r \cdot m_1$, and $v = s \cdot m_2$. Hence, we can continue our derivation above as,

$$
\begin{aligned}
&\frac{1}{c}\left[(ux+c)(vx+c)\right] \\
&= \frac{1}{m_1 m_2}(rm_1 \cdot x + m_1 m_2)(sm_2 \cdot x + m_1 m_2) \\
&= \frac{1}{m_1}(rm_1 \cdot x + m_1 m_2) \cdot \frac{1}{m_2}(sm_2 \cdot x + m_1 m_2) \\
&= \left(\frac{rm_1}{m_1} \cdot x + \frac{m_1 m_2}{m_1}\right)\left(\frac{sm_2}{m_2} \cdot x + \frac{m_1 m_2}{m_2}\right) \\
&= (rx + m_2)(sx + m_1)
\end{aligned}
$$

We have now covered two steps of what I consider a three-step argument. In the first step, we showed that if integers u and v exists such that $u + v = b$ and $uv = ac$, then that pair is unique. We can always switch between which integer we call u and which integer we call v, but the pair itself is one of a kind. In the second step, we showed that if our integers u and v exist, then the trinomial can indeed be factored. And in our final third step, we are going to show that if the integers u and v exist, then we can factor the trinomial using the particular method of factoring by grouping that we illustrated at the beginning. Let's get into it!

Now if our trinomial can be factored into a product of two binomials with integer coefficients, then that means we can write,

$$ax^2 + bx + c = (p_1 x + p_2)(p_3 x + p_4)$$

If we multiply out the right hand side of this equation a bit, then we can rewrite it as,

$$(p_1 x + p_2)(p_3 x + p_4) = p_1 p_3 x^2 + p_2 p_3 x + p_1 p_4 x + p_2 p_4$$

But notice that in this instance $p_2 p_3 + p_1 p_4 = b$ and $p_2 p_3 \cdot p_1 p_4 = p_1 p_3 \cdot p_2 p_4 = ac$. Hence, we can set $u = p_2 p_3$ and $v = p_1 p_4$, and remember that this is the only pair of integers which will now solve the equations $u + v = b$ and $uv = ac$. Again, it doesn't matter which of these products we call u and which one we call v. It will still be the same pair in one order or the other, and we can proceed as follows.

$$ax^2 + bx + c$$

$$= ax^2 + ux + vx + c$$

$$= p_1 p_3 x^2 + p_2 p_3 x + p_1 p_4 x + p_2 p_4$$

$$= (p_1 p_3 x^2 + p_2 p_3 x) + (p_1 p_4 x + p_2 p_4)$$

$$= p_3 x(p_1 x + p_2) + p_4 (p_1 x + p_2)$$

$$= (p_3 x + p_4)(p_1 x + p_2)$$

And we’re done!