HOMORMORPHISMS AND ONE-TO-ONE FUNCTTIONS

Discussion: The theorem below give a very useful result. It shows that if we have a homomorphism from one group onto another, then another way to show that this homomorphism is also a one-to-one function is to simply verify that the only element in the Kernel is the identity.

Theorem: Let $f : A \to B$ be a homomorphism from a group *A* onto a group *B*. Then *f* is one-to-one if and only if $Ker(f) = \{e\}$.

Proof: Suppose $f : A \to B$ is a homomorphism from a group *A* onto a group *B*, and suppose that *f* is one-to-one. By previous proof, we know that $f(e) = e$, and if *f* is one-to-one, then it follows that $Ker(f)$ contains only the identity, *e*.

Now suppose that $Ker(f) = \{e\}$, and suppose that *f* is not one-to-one. Then there exists $a, b \in A$ with $a \neq b$ such that $f(a) = f(b)$. But this means that $e = f(a)f(b)^{-1} = f(a)f(b^{-1}) = f(ab^{-1})$ where $ab^{-1} \neq e$. But this contradicts our assumption that $Ker(f) = \{e\}$. Therefore, *f* is one-to-one.

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