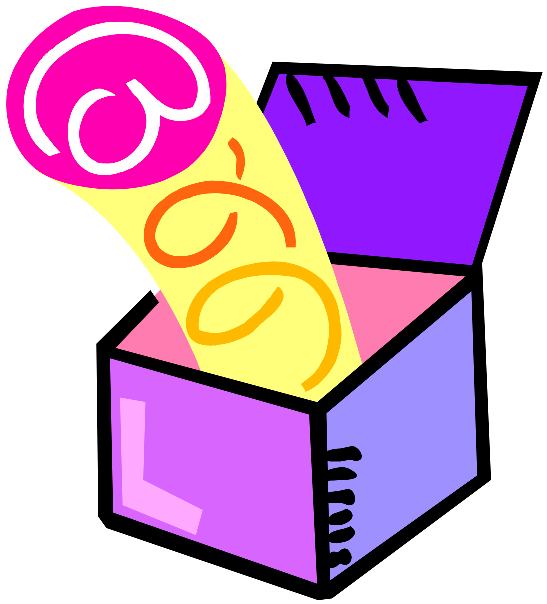
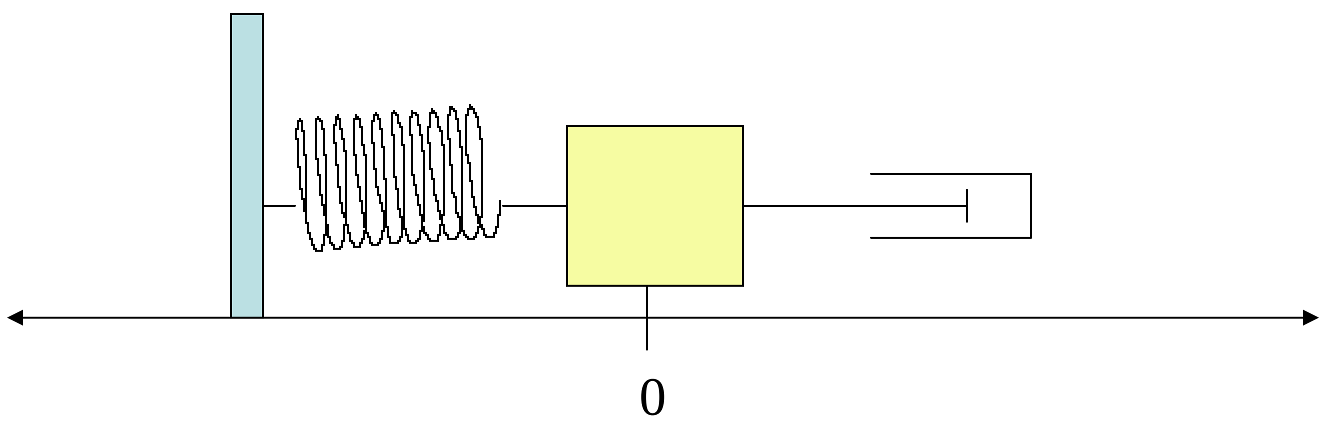
The Incredible Harmonic Oscillator!

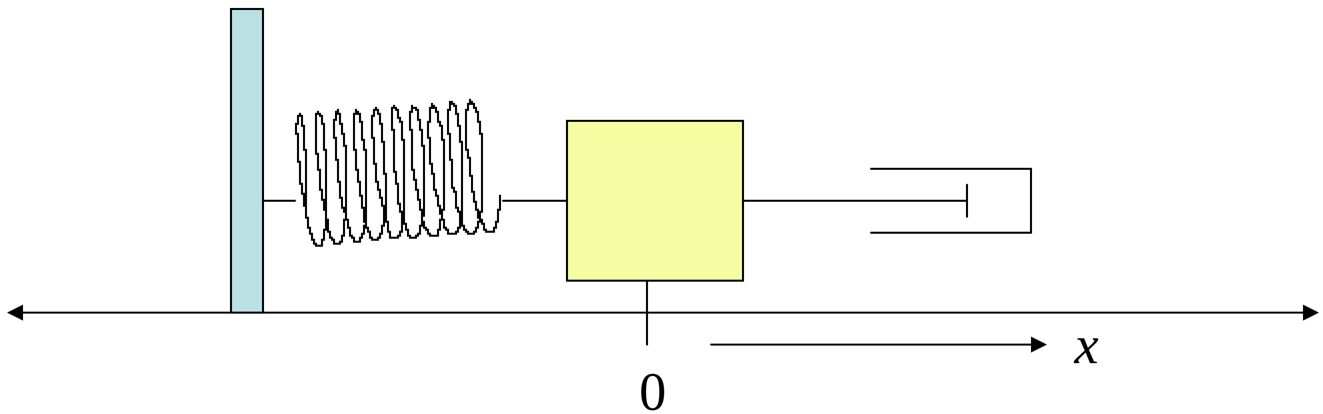


$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$$

Suppose we have a spring, a mass, and a dashpot.

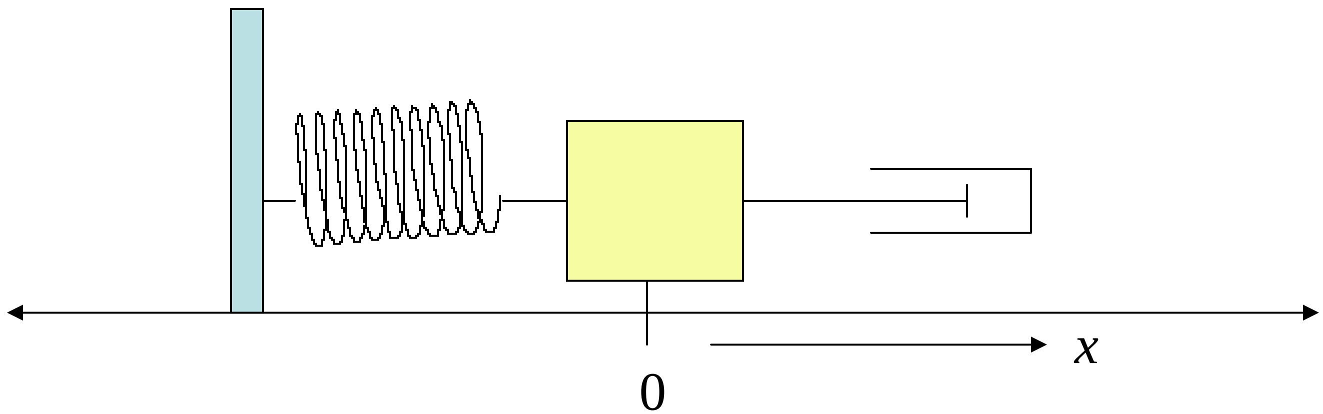


Then the force on the mass is determined both by Hooke's law and the damping effect of the dashpot.



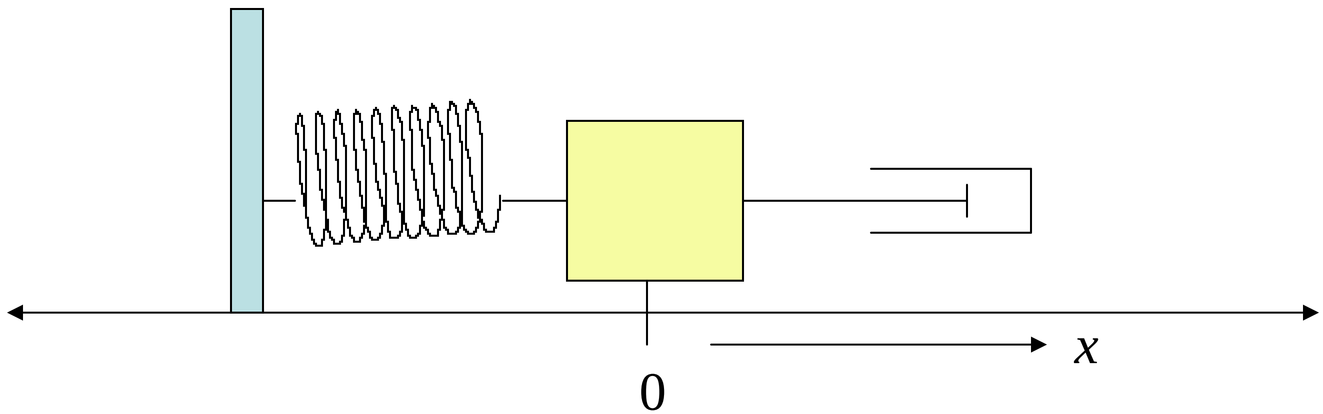
$$F = -kx - c\frac{dx}{dt} \quad (c > 0)$$

Hooke's law says that the force is proportional to the displacement of the mass, and the damping effect is proportional to the velocity of the mass.



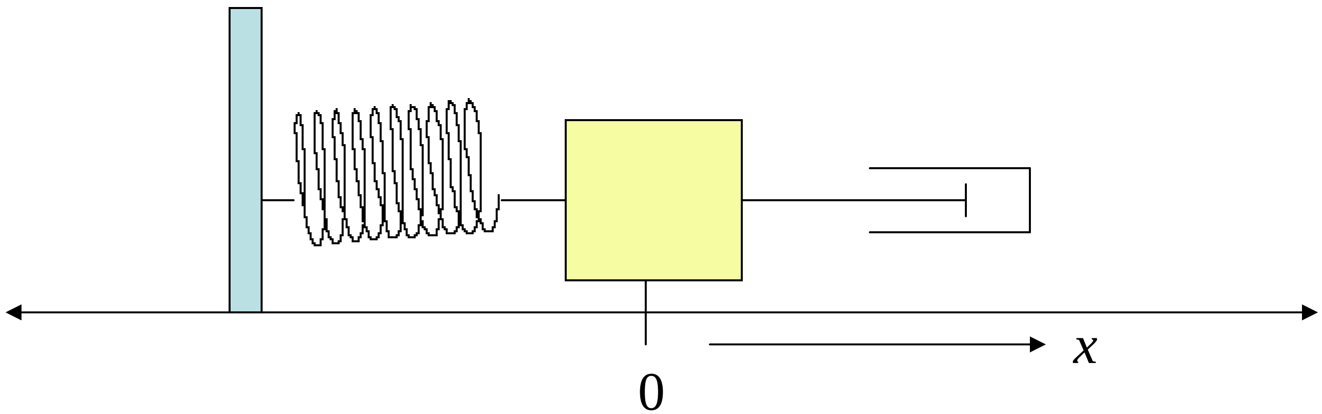
$$F = -kx - c\frac{dx}{dt} \quad (c > 0)$$

Furthermore, Newton's Second Law of Motion tells us that *force=mass x acceleration.* However, acceleration is the second derivative of position, and so we can write the left-hand side of the equation as follows:



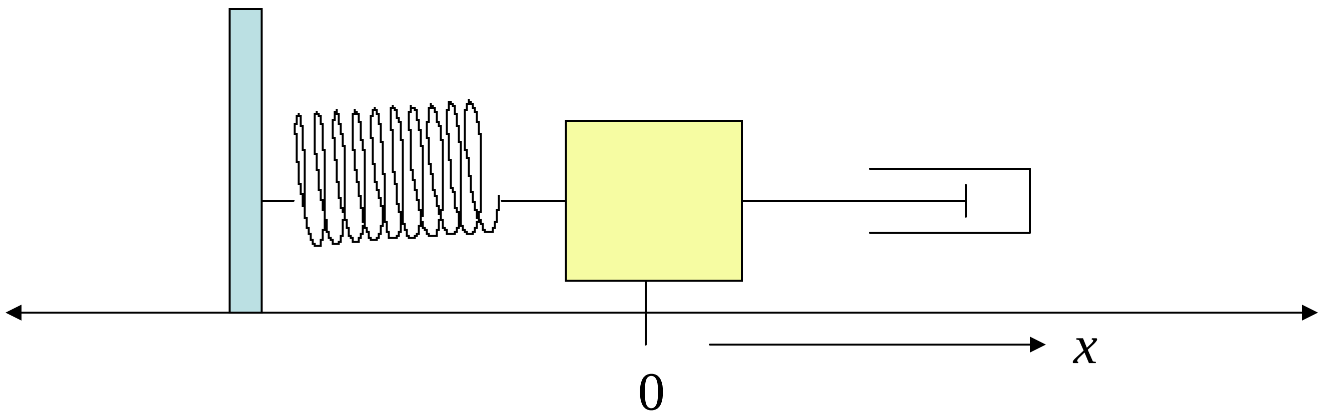
$$m\frac{d^2x}{dt^2} = -kx - c\frac{dx}{dt}$$

The end result is a second degree homogeneous linear differential equation with constant coefficients.



$$m\frac{d^2x}{dt^2}+c\frac{dx}{dt}+kx=0$$

If we now divide everything by *m*, then we can write our harmonic oscillator as follows.



$$\frac{d^2x}{dt^2} + p\frac{dx}{dt} + qx = 0$$

So now let's review how we might solve a typical problem of this sort.

$$\frac{d^2 y}{dt^2} + 3\frac{dy}{dt} + 2y = 0$$

The matrix method involves turning it first into a system of linear differential equations.

$$\frac{dy}{dt} = v$$

$$\frac{dv}{dt} = -2y - 3v$$

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = 0$$

We now write the system as a matrix equation.

$$\frac{dY}{dt} = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} \begin{pmatrix} y \\ v \end{pmatrix}$$

$$\frac{dy}{dt} = v$$

$$\frac{dv}{dt} = -2y - 3v$$

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = 0$$

The next step is to find the characteristic polynomial, set it equal to zero, and solve for the eigen values.

$$A = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}$$

$$\left| A - \lambda I \right| = \begin{vmatrix} -\lambda & 1 \\ -2 & -3-\lambda \end{vmatrix} = \lambda^2 + 3\lambda + 2 = (\lambda+1)(\lambda+2) = 0$$

$$\Rightarrow \lambda = -1 \text{ or } \lambda = -2$$

$$\frac{dy}{dt} = v$$

$$\frac{dv}{dt} = -2y - 3v$$

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = 0$$

Next, we need to find eigenvectors that are associated with each eigenvalue.

$$\lambda = -1$$

$$AV_1 = -V_1 \Rightarrow \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}\begin{pmatrix} y_1 \\ v_1 \end{pmatrix} = \begin{pmatrix} -y_1 \\ -v_1 \end{pmatrix} \Rightarrow \begin{aligned} v_1 &= -y_1 \\ -2y_1 - 3v_1 &= -v_1 \end{aligned}$$

$$\frac{dy}{dt} = v$$

$$\frac{dv}{dt} = -2y - 3v$$

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = 0$$

The information provided by the second equation is redundant, and we just need to make sure that our coordinates have opposite signs.

$$\lambda = -1$$

$$AV_1 = -V \Rightarrow \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} \begin{pmatrix} y_1 \\ v_1 \end{pmatrix} = \begin{pmatrix} -y_1 \\ -v_1 \end{pmatrix} \Rightarrow \begin{array}{r} v_1 = -y_1 \\ -2y_1 - 3v_1 = -v_1 \end{array}$$

$$\text{Let } V_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\frac{dy}{dt} = v$$

$$\frac{dv}{dt} = -2y - 3v$$

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = 0$$

We now find an eigenvector for our second eigenvalue.

$$\lambda = -2$$

$$AV_2 = -2V_2 \Rightarrow \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}\begin{pmatrix} y_2 \\ v_2 \end{pmatrix} = \begin{pmatrix} -2y_2 \\ -2v_2 \end{pmatrix} \Rightarrow \begin{aligned} v_2 &= -2y_2 \\ -2y_2 - 3v_2 &= -2v_2 \end{aligned}$$

$$\text{Let } V_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$\frac{dy}{dt} = v$$

$$\frac{dv}{dt} = -2y - 3v$$

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = 0$$

We can now write down the solution to our system.

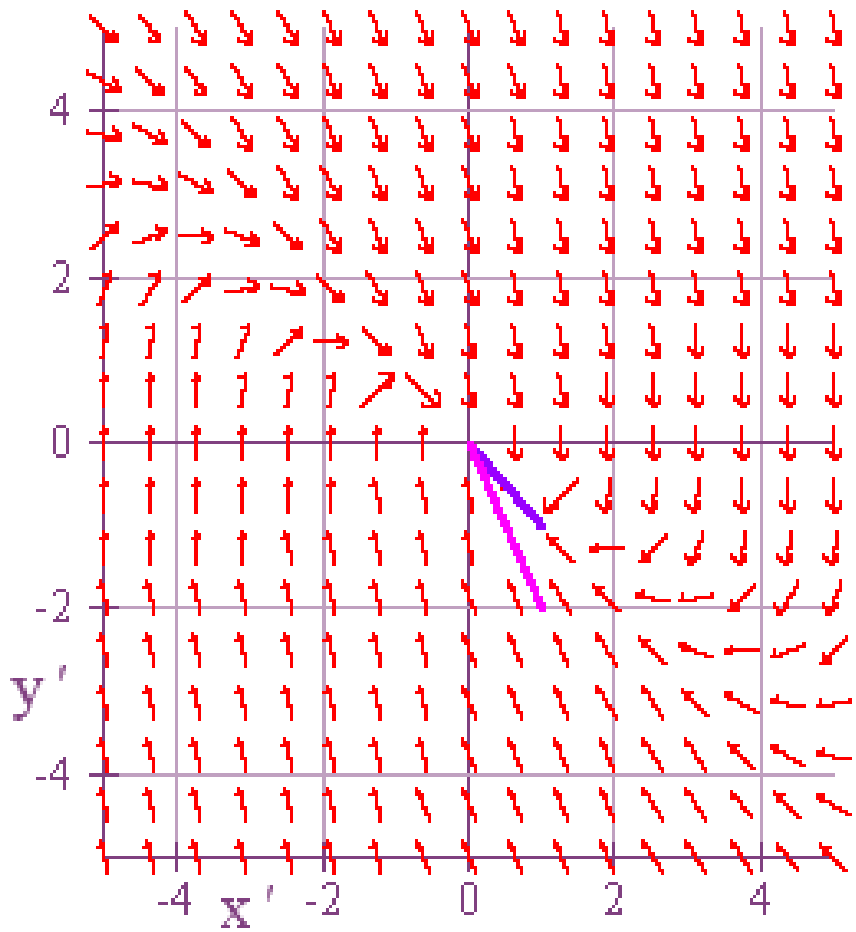
$$Y(t) = k_1 e^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + k_2 e^{-2t}\begin{pmatrix} 1 \\ -2 \end{pmatrix} \Rightarrow \begin{aligned} y &= k_1 e^{-t} + k_2 e^{-2t} \\ \frac{dy}{dt} &= -k_1 e^{-t} - 2k_2 e^{-2t} \end{aligned}$$

$$\frac{dy}{dt} = v$$

$$\frac{dv}{dt} = -2y - 3v$$

$$\frac{d^2 y}{dt^2} + 3\frac{dy}{dt} + 2y = 0$$

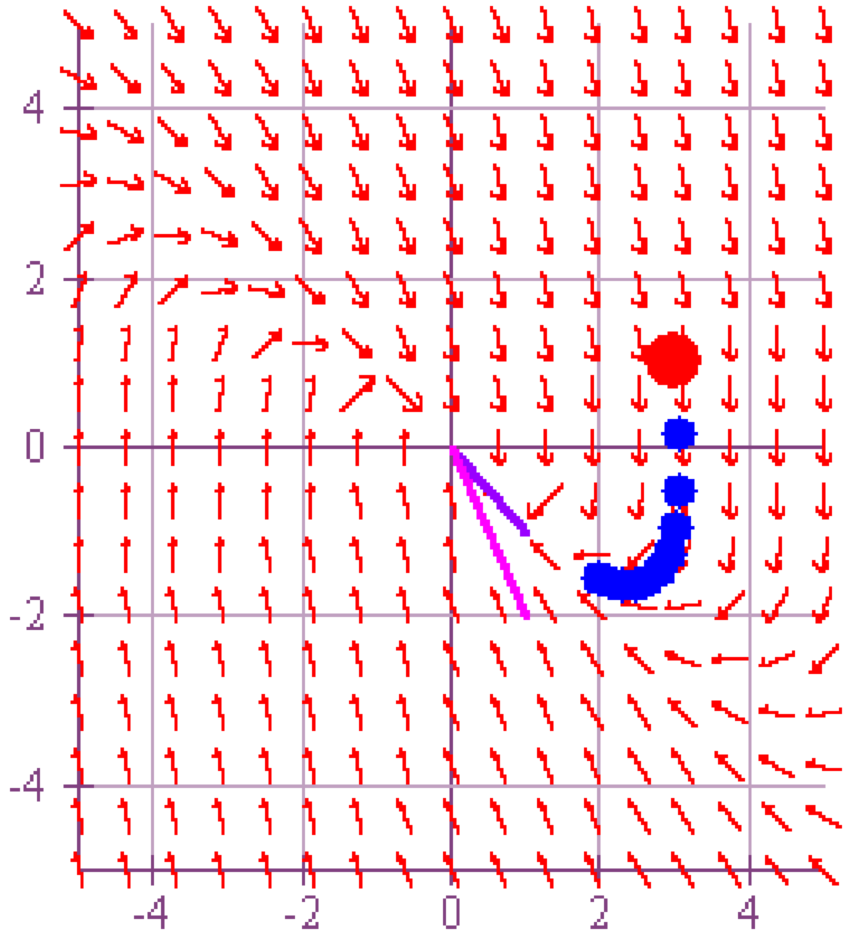
And here is what the phase portrait with eigenvectors looks like.



$$\frac{dy}{dt} = v$$

$$\frac{dv}{dt} = -2y - 3v$$

Now let's take an initial condition such as *y=3* and *v=1* when *t=0*.



$$\frac{dy}{dt} = v$$

$$\frac{dv}{dt} = -2y - 3$$

That means we need to solve the following system of equations.

$$y = k_1 e^{-t} + k_2 e^{-2t}$$

$$\frac{dy}{dt} = -k_1 e^{-t} - 2k_2 e^{-2t}$$

$$k_1 + k_2 = 3$$

$$-k_1 - 2k_2 = 1$$

$$k_1 = 7$$

$$k_2 = -4$$

$$\frac{dy}{dt} = v$$

$$\frac{dv}{dt} = -2y - 3v$$

And here is the solution to our initial value problem.

$$Y(t) = -e^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + 4e^{-2t}\begin{pmatrix} 1 \\ -2 \end{pmatrix} \Rightarrow \begin{array}{l} y = 7e^{-t} - 4e^{-2t} \\ \dfrac{dy}{dt} = -7e^{-t} + 8e^{-2t} \end{array}$$

$$\frac{dy}{dt} = v$$

$$\frac{dv}{dt} = -2y - 3v$$

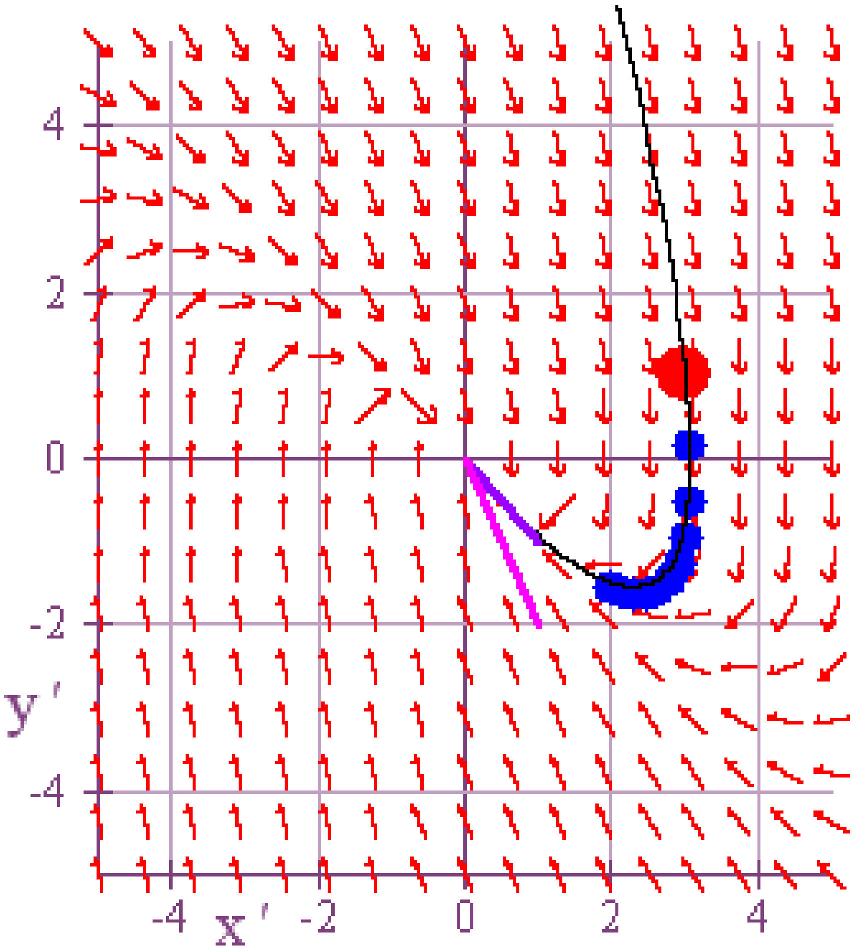
$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = 0$$

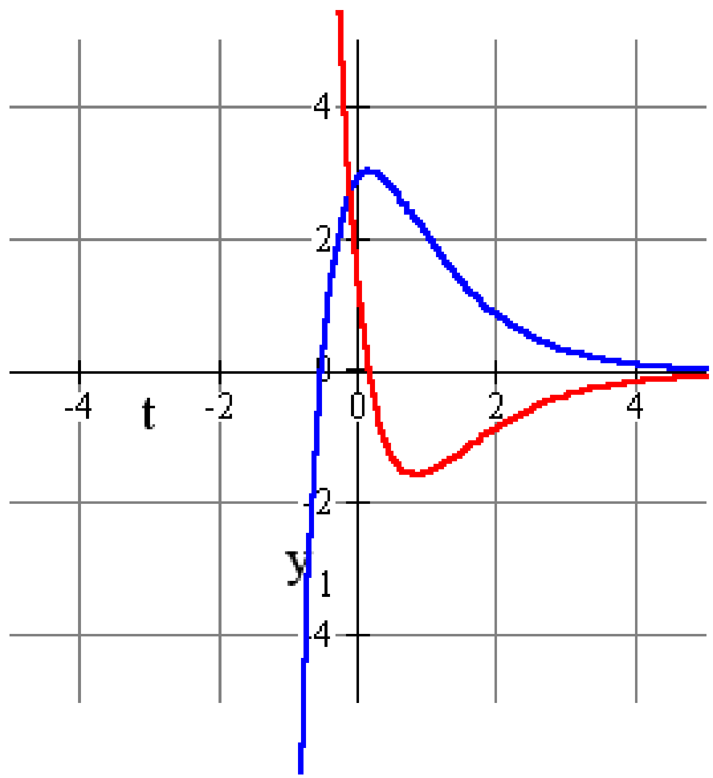
$$t = 0, y = 3, \frac{dy}{dt} = 1$$

Let’s now check some graphs!

$$y = 7e^{-t} - 4e^{-2t}$$

$$\frac{dy}{dt} = -7e^{-t} + 8e^{-2t}$$





QUESTION: Is there an easier way to solve our harmonic oscillator?

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = 0$$

Yes! Just assume that the solution has the form $y = e^{rx}$.

$$r^2 e^{rx} + 3re^{rx} + 2e^{rx} = 0$$

$$\Rightarrow (r^2 + 3r + 2)e^{rx} = 0$$

$$\Rightarrow r^2 + 3r + 2 = 0$$

$$\Rightarrow (r+1)(r+2) = 0$$

$$\Rightarrow r = -1 \text{ or } r = -2$$

$$\frac{d^2 y}{dt^2} + 3\frac{dy}{dt} + 2y = 0$$

Notice that we wound up with the same characteristic polynomial as before.

$$r^2 e^{rx} + 3re^{rx} + 2e^{rx} = 0$$

$$\Rightarrow (r^2 + 3r + 2)e^{rx} = 0$$

$$\Rightarrow r^2 + 3r + 2 = 0$$

$$\Rightarrow (r+1)(r+2) = 0$$

$$\Rightarrow r = -1 \text{ or } r = -2$$

$$\frac{d^2 y}{dt^2} + 3\frac{dy}{dt} + 2y = 0$$

And we can now write down the solution as follows.

$$y = k_1 e^{-t} + k_2 e^{-2t}$$

$$\frac{dy}{dt} = -k_1 e^{-t} - 2k_2 e^{-2t} \Rightarrow k_1 e^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + k_2 e^{-2t}\begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$\frac{d^2 y}{dt^2} + 3\frac{dy}{dt} + 2y = 0$$

And lastly, notice that our final form shows both the eigenvalues and some eigenvectors for the corresponding system of equations.

$$\begin{aligned} y &= k_1 e^{-t} + k_2 e^{-2t} \\ \frac{dy}{dt} &= -k_1 e^{-t} - 2k_2 e^{-2t} \end{aligned} \Rightarrow k_1 e^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + k_2 e^{-2t}\begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$\frac{d^2 y}{dt^2} + 3\frac{dy}{dt} + 2y = 0$$

And the solution to the initial value problem is the same.

$$y = 7e^{-t} - 4e^{-2t}$$

$$\frac{dy}{dt} = -7e^{-t} + 8e^{-2t}$$

$$y = k_1 e^{-t} + k_2 e^{-2t}$$

$$\frac{dy}{dt} = -k_1 e^{-t} - 2k_2 e^{-2t} \Rightarrow k_1 e^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + k_2 e^{-2t}\begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = 0$$

$$t = 0, y = 3, \frac{dy}{dt} = 1$$

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = 0$$

$$t = 0, y = 3, \frac{dy}{dt} = 1$$

COOL!

$$y = 7e^{-t} - 4e^{-2t}$$

$$\frac{dy}{dt} = -7e^{-t} + 8e^{-2t}$$