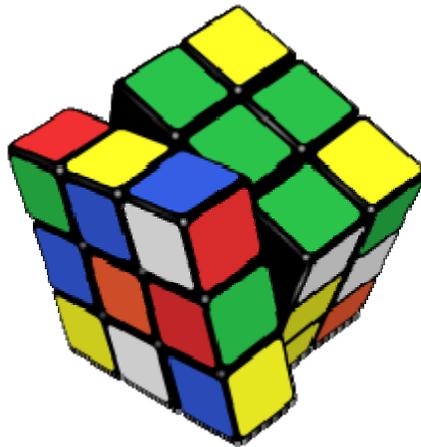


## Lesson 6

### GENERATOR DIAGRAMS

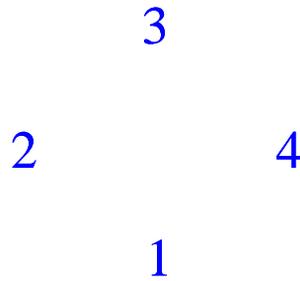
We've talked before about the concept of a group acting on the elements of a set, and these days there is no better example of that than Rubik's cube. Rubik's cube is a fascinating puzzle that was invented in 1974 by a Hungarian sculptor and professor of architecture named Ernös Rubik, but it wasn't until 1980 that the puzzle began to be marketed in the United States by Ideal Toy Corporation and, subsequently, became widely popular. The puzzle itself is deceptively simple in appearance. You have a cube with six faces, and each face of the cube is divided into several smaller cubes, and then the faces themselves can be rotated in several directions in order to create an almost unfathomable number of permutations of the colored squares on each little cube. Many a person has spent many an hour trying to figure out how to unscramble their cube only to simply take it apart with a screwdriver and then reassemble it!



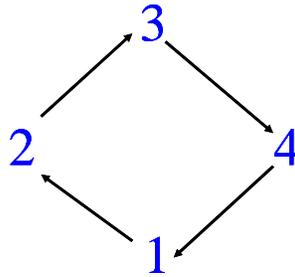
When we look at the cube, we quickly realize that there are six basic moves that we can perform on the cube, and one usually denotes these moves by the letters  $R$ ,  $L$ ,  $U$ ,  $D$ ,  $F$ , and  $B$ . These moves represent making quarter-turns in the clockwise direction, respectively, of the right face, left face, up face, down face, front face, and back space of the cube. Some people, however, like to write these letters in the order  $BFUDLR$  so that it will appropriately be pronounced “befuddler.” But for our purposes, the main thing to realize is that the set of all possible expressions of finite length that can be made from the letters  $R$ ,  $L$ ,  $U$ ,  $D$ , and  $F$  form a group where the group multiplication is simply one expression followed by another. Furthermore, this group acts on the 26 cubelets and 54 facelets of the cube to produce 43,252,003,274,489,856,000 distinct permutations.

Notice that if we think of each cubelet or facelet as being represented by a point, then the permitted moves that generate Rubik's cube simply move these points around in various ways. Well, forty-three quintillion permutations is a little bit to much for us to study easily, so we're going to make the process simpler by taking a very small number of points combined with moves that will generate groups of much smaller size. We'll start with four points that we'll depict by just the labels 1, 2, 3, and 4.

## Lesson 6



We've seen before that with four objects you can make  $4! = (4)(3)(2)(1) = 24$  permutations, and thus the size of the symmetric group of degree 4, i.e.  $S_4$  the group of all permutations that can be made of four objects, is  $|S_4| = 24$ . From this we should realize that any specific moves we define to move our number labels around will result in a group that is a subgroup of  $S_4$ . Hence, the order of our group must divide 24. For example, consider the following move which in cycle notation can be written as  $(1,2,3,4)$ .

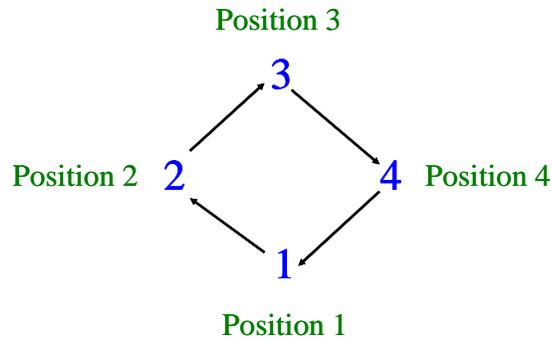


It should be clear in this case that our maneuver is just going to create a cyclic group of order 4, and sure enough, 4 divides evenly into 24.

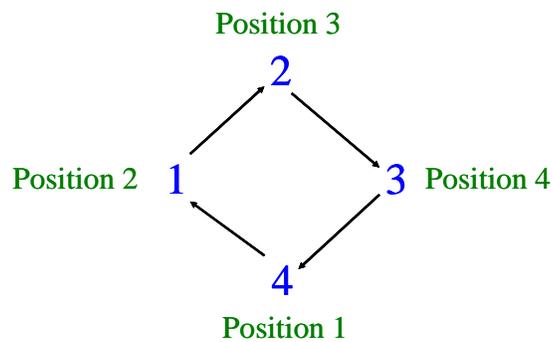
$$\{ (), (1,2,3,4), (1,3)(2,4), (1,4,3,2) \}$$

By the way, a good way to think about a permutation such as  $(1,2,3,4)$ , when we are dealing with diagrams such as the one above which I call a “generator diagram,” is that it means move the number or object that is in position 1 to position 2, move what is in position 2 to position 3, move what is in position 3 to position 4, and finally move what is in position 4 into position 1. It doesn't mean replace the number 1 at the bottom with the number 2 at the bottom.

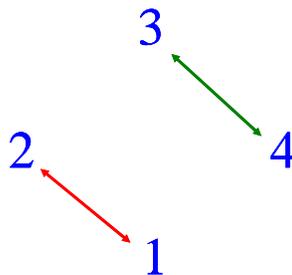
## Lesson 6



Hence, when we apply  $(1,2,3,4)$  to the above diagram, the numbers in blue move to the following positions.



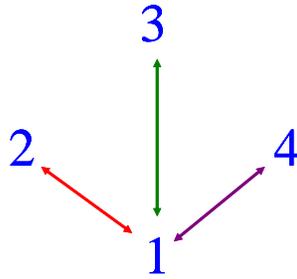
As another example of a generator diagram, the set of two moves illustrated by the diagram below is analogous to having two light switches that you can independently switch on and off. Thus, the corresponding group generated will have order 4 and is the one that we previously identified as the Klein 4-group.



$$\{ (), (3,4), (1,2), (1,2)(3,4) \}$$

Now consider the diagram below. What size group do you think these moves will generate?

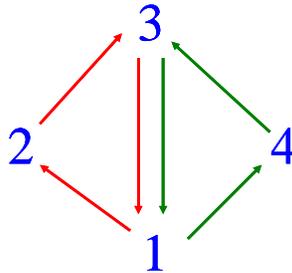
## Lesson 6



Well, it turns out that the permutations  $(1,2)$  &  $(1,3)$  &  $(1,4)$  generate the entire group,  $S_4$ , of all 24 permutations that can be made of four objects.

{  $()$ ,  $(3,4)$ ,  $(2,3)$ ,  $(2,3,4)$ ,  $(2,4,3)$ ,  $(2,4)$ ,  $(1,2)$ ,  $(1,2)(3,4)$ ,  $(1,2,3)$ ,  $(1,2,3,4)$ ,  $(1,2,4,3)$ ,  $(1,2,4)$ ,  $(1,3,2)$ ,  $(1,3,4,2)$ ,  $(1,3)$ ,  $(1,3,4)$ ,  $(1,3)(2,4)$ ,  $(1,3,2,4)$ ,  $(1,4,3,2)$ ,  $(1,4,2)$ ,  $(1,4,3)$ ,  $(1,4)$ ,  $(1,4,2,3)$ ,  $(1,4)(2,3)$  }

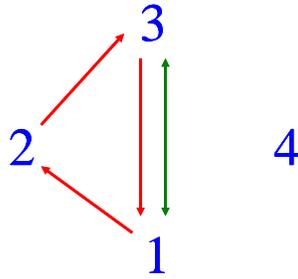
Now let's look at the group generated by the permutations  $(1,2,3)$  &  $(1,4,3)$ . This will generate a subgroup of  $S_4$  that has 12 elements, and closer examination suggests that this is  $A_4$ , the alternating group of degree 4 that consists of all the even permutations in  $S_4$ .



{  $()$ ,  $(2,3,4)$ ,  $(2,4,3)$ ,  $(1,2)(3,4)$ ,  $(1,2,3)$ ,  $(1,2,4)$ ,  $(1,3,2)$ ,  $(1,3,4)$ ,  $(1,3)(2,4)$ ,  $(1,4,2)$ ,  $(1,4,3)$ ,  $(1,4)(2,3)$  }

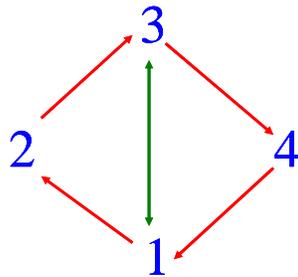
On the other hand, the subgroup generated by these moves below will contain only 6 elements, and it will be isomorphic to  $D_3$  and  $S_3$ . Notice that both our moves leave the fourth point unchanged.

## Lesson 6



$$\{ (), (2,3), (1,2), (1,2,3), (1,3,2), (1,3) \}$$

Similarly, the moves for the following diagram will generate a group of order 8 that is isomorphic to  $D_4$ .



$$\{ (), (2,4), (1,2)(3,4), (1,2,3,4), (1,3), (1,3)(2,4), (1,4,3,2), (1,4)(2,3) \}$$

Some good games to play with these kinds of diagrams, which again I call generator diagrams, are (1) find a sequence of moves that will change the diagram from one configuration into another, (2) find the number of elements in the group generated by the indicated moves, and (3) given a set of points, find different sets of moves that generate the same group.