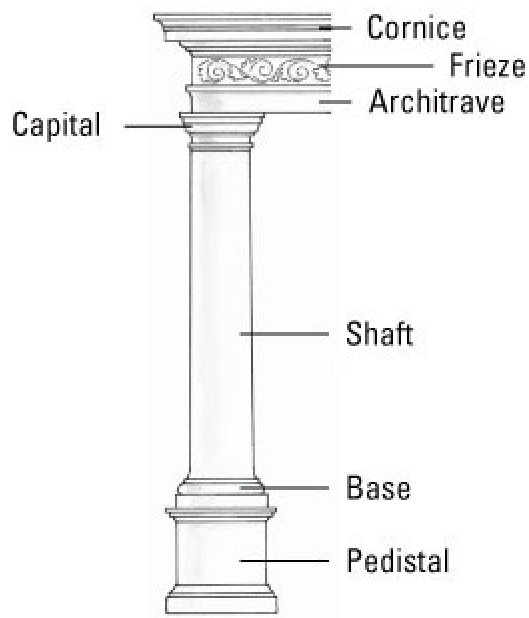
Lesson 20

FRIEZE GROUPS

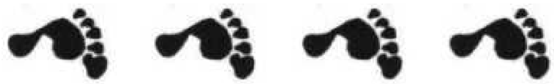
Early on we mentioned that groups measure symmetry and that wherever we find symmetry present, we also find a corresponding group. We've also stated that symmetry is just the repetition of a pattern in some form or fashion, and that the maneuvers that convert one instance of that pattern into another combine to form the elements of the associated group. We're now going to return to symmetry, and we begin with an introduction to frieze groups.

A frieze is the decorative strip that may adorn the edge of a column or building, and it is most often associated with Greek architecture.

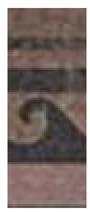


Mathematicians have studied and classified the various types of symmetrical patterns that can occur in a frieze, and, as a result, seven different patterns have been identified, and they all involve a mixture of translations, reflections, and rotations. In this lesson we are going to explain and give examples of each of the seven patterns. Also, the notation that was originally developed by mathematicians for these patterns was anything but intuitive. Thus, I am going to use a more modern and self-explanatory notation introduced by the British mathematician, John Conway.

<u>F1-HOP:</u> The *hop* is a frieze pattern that is generated solely by translation without any reflections or rotations. Below is a characterization of this pattern using a footstep along with a real world example.



In classifying a frieze pattern, we need to be aware that they all contain symmetry or repetition based upon the horizontal movement or translation of a particular motif, and that we also need to examine the image for reflections and rotational symmetries. And we also want to identify a smallest possible piece of our frieze that could be used to generate the whole image. In the above example, it appears that the entire image can be translated by shifting the image in the cell below either left or right.



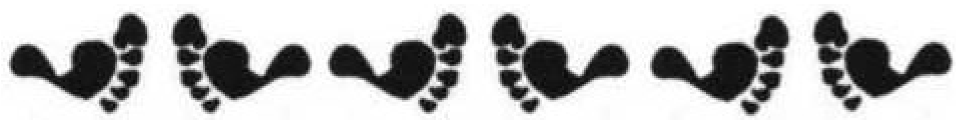
In this case, the corresponding group is going to isomorphic to the integers under addition, $(\mathbb{Z},+)$. In particular, in this group 0 corresponds to no movement at all, 1 corresponds to moving one unit to the right, and -1 represents moving the image one unit to the left.

<u>F2-STEP:</u> The *step* is also known as a glide reflection. This is a type of reflection that begins with a translation that is immediately followed by a reflection, usually across a horizontal line.

To generate the first image, we can think of taking a single foot print,  ,

sliding it to the right, and then reflecting it to get our next foot print,  . The group created is also isomorphic to the integers under addition, $(\mathbb{Z},+)$, where 1 means "move to the right and flip" while -1 means "move to the left and flip," but in spite of getting back the same group associated with F1, the patterns, F1 and F2, are still distinct since F1 involves only a translation while F2 involves both a translation and a flip.

<u>F3-SIDLE:</u> The word *sidle* can mean to walk sideways, and the first image below looks like what would happen if you stood with your feet turned out and then tried to walk sideways.



The *sidle* contains both a translation and a reflection about what is normally a vertical axis. If you look at the first image above, you can see how to create F3, a *sidle*. You can start with the imprint of a single right foot, and then reflect it about a vertical axis. We usually think of using the same, fixed vertical axis for all reflections. Next, just translate both foot prints left or right to complete the frieze. The presence of both translations and vertical reflections in the second image is an immediate giveaway that the symmetry pattern is F3, the *sidle*.

<u>F4-SPINNING HOP:</u> The *spinning hop* combines a translation with a rotation through $180°$. Notice that you can also accomplish this half-turn by doing a reflection about a horizontal axis that is immediately followed by a reflection across a vertical axis. In the image below, you can start with a single footprint with the toes on the left, reflect it about a horizontal and then a vertical axis to get a second footprint, and then translate both feet right or left to complete things.

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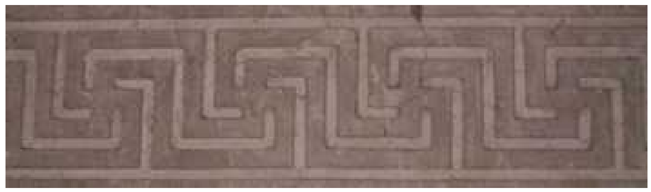
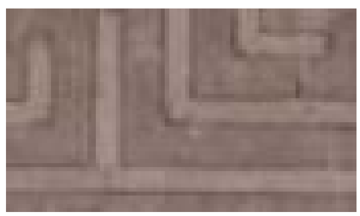
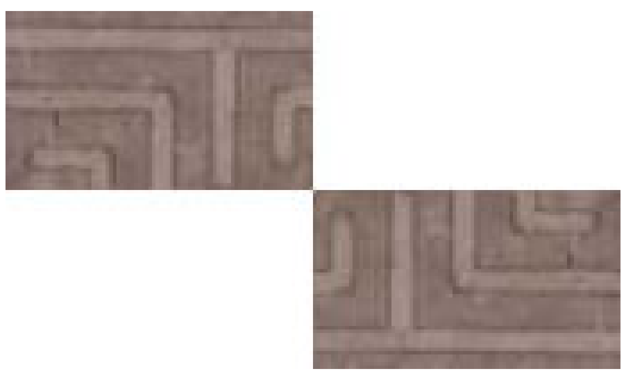


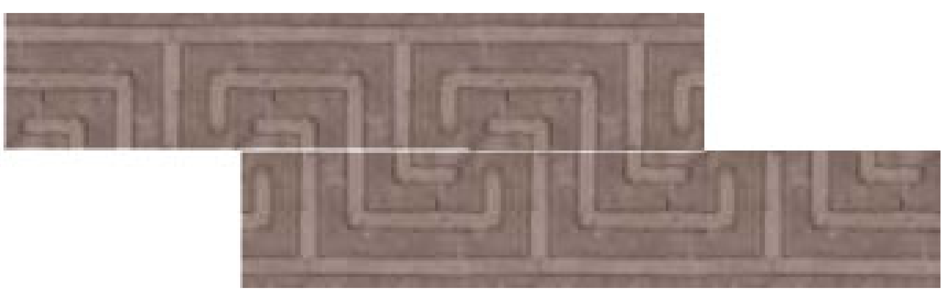
To generate this image, I can start with a rectangle that extends from one center of rotation to the next.



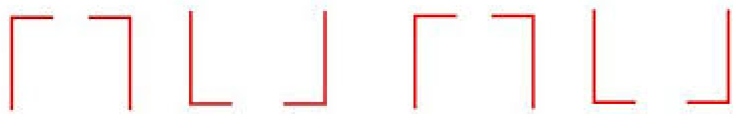
Next, we rotate this piece $180°$ about the upper left corner.



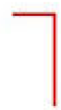
And now, we can begin to complete the image by doing translations right and left.



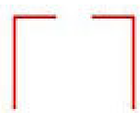
<u>F5-SPINNING SIDLE:</u> This particular frieze pattern, the *spinning sidle*, exhibits translations, glide reflections, reflections about a vertical axis, and rotations through 180°.



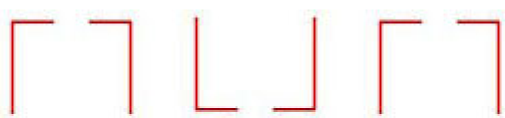
Even though this frieze pattern exhibits multiple symmetries, the whole frieze can be generated by taking an image representing the fundamental pattern and then applying reflections about a vertical axis and glide reflections as well. For example, we can start with the following image.



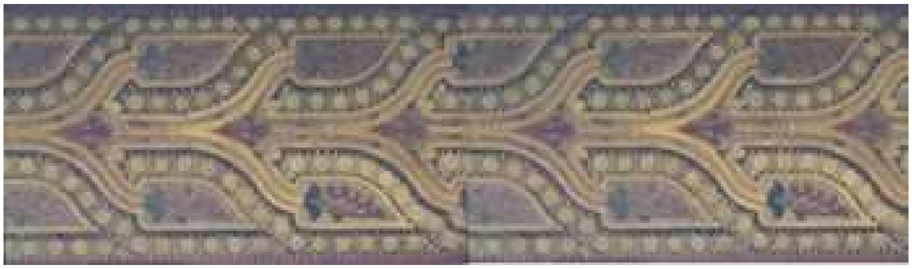
Now reflect this image about a vertical axis.



And lastly, do glide reflections.



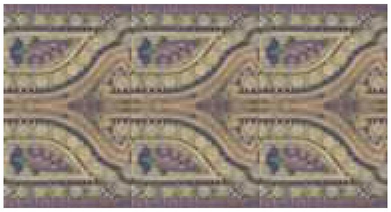
<u>F6-JUMP:</u> This pattern, the *jump*, is exactly what you would get if you stood on two feet and hopped in a consistent direction. The image consists of a translation coupled with a horizontal reflection.



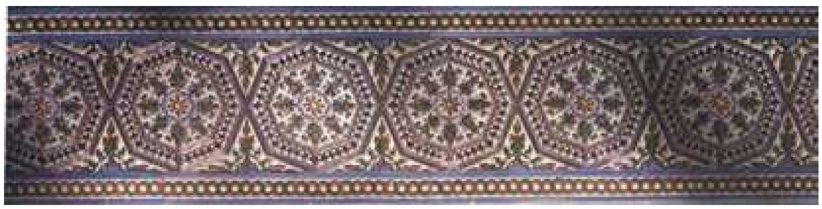
We can generate this type of frieze by starting first with a fundamental region.

Next, we reflect this piece about a horizontal axis.

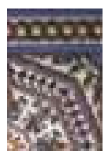
And to complete the image, we do a translation of this image right and left.



<u>F7-SPINNING JUMP:</u> The *spinning jump* combines all possible maneuvers. In other words, it contains translations, glide reflections, vertical reflections, rotations, and horizontal reflections. However, the image itself can be generated by applying just translations, vertical reflections, and horizontal reflections to a fundamental region.



So, let's start with a fundamental image that can be used to generate the rest of the pattern.



Do a flip about a vertical axis.

Follow this with a flip about a horizontal axis.

And finally, translate this image left and right to finish generating the frieze.

By the way, in the image below, you might have also noticed the presence of rotational symmetry through angles of 45°, and you can rightly wonder why this symmetry isn't part of our analysis. The answer is because in the symmetries that we associate with frieze groups, we restrict ourselves to rigid motions of the plan. For example, if we rotated the entire image through angles that are integer multiples of 45°, then we could get something like the following which deviates from our frieze pattern.

And finally, below is a nice flowchart that can help you identify which of the seven types of frieze groups you are observing. Enjoy!