Exact Differential Equations

$$P(x,y)+Q(x,y)\frac{dy}{dx}=0$$

If there is a function z=f(x,y) with continuous partial derivatives such that z_x=P and z_y=Q, then the differential equation is exact with general solution f(x,y)=C.

$$P(x,y)+Q(x,y)\frac{dy}{dx}=0$$

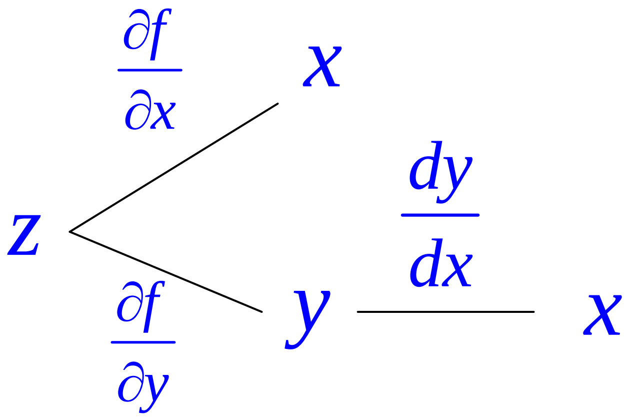
$$\text{Exact if } \frac{\partial f}{\partial x}=P \;\;\&\;\; \frac{\partial f}{\partial y}=Q.$$

If P and Q have continuous partial derivatives on an open disk R, then the equation below is exact if and only if $P_y = Q_x$.

$$P(x, y) + Q(x, y)\frac{dy}{dx} = 0$$

$$\text{Exact if } \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

If z=f(x,y) and y=y(x), then we can derive the left-hand side of the equation below from a chain rule.



$$\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = P(x,y) + Q(x,y)\frac{dy}{dx} = 0$$

Another way to think about it is to rewrite our equation in differential form. This makes the left-hand side look like part of a line integral for computing work done, and if $P_y = Q_x$, then a potential function $z = f(x,y)$ exists.

$$P(x,y) + Q(x,y)\frac{dy}{dx} = 0$$

$$\Rightarrow P(x,y)dx + Q(x,y)dy = 0$$

Recall if $\vec{F} = P\,\hat{i} + Q\,\hat{j}$, then $\int_C \vec{F} \bullet d\vec{r} = \int_C (P\,dx + Q\,dy)$.

The procedure for finding z=f(x,y) for an exact equation is the same as that for finding a potential function for a work integral.

$$P(x,y)+Q(x,y)\frac{dy}{dx}=0$$

$$\Rightarrow P(x,y)dx+Q(x,y)dy=0$$

$$\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}$$

Example:

$$2x^3y^2 + x^4y\frac{dy}{dx} = 0$$

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Thus, the general solution of:

$$2x^3y^2 + x^4y\frac{dy}{dx} = 0$$

is:

$$\frac{x^4y^2}{2} = C$$

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$$\sin y - (y - x\cos y)\frac{dy}{dx} = 0$$

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$$x\sin y - \frac{y^2}{2} = C$$

As a final note, all separable differential equatations are exact.

$$P(x) + Q(y)\frac{dy}{dx} = 0$$

$$\frac{\partial P}{\partial y} = 0 = \frac{\partial Q}{\partial x}$$