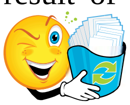
EVERYTHING ELSE THAT I FORGOT TO MENTION – PRACTICE

If A and B are sets, then the *Cartesian product* of A and B is defined as $A \times B = \{(a,b) \mid a \in A \text{ and } b \in B\}$. In other words, the Cartesian product of the two sets is the set of all ordered pairs that can be formed by pairing elements of the first set with elements of the second set. For example, using this definition, we can think of the coordinate plane as just the Cartesian product of the real numbers with the real numbers, $\mathbb{R} \times \mathbb{R}$. Additionally, this construction is going to allow us to give a more abstract definition of things we are already familiar with such as *functions*.

Anytime you see something in mathematics called *Cartesian*, you know it is being named after René Descartes (1596 – 1650). During his lifetime, he was famous both as a philosopher and as a mathematician. He also worked as a mercenary soldier. It was a time when mathematicians were more like Rambo. Furthermore, he died at age 53 as a result of getting up too early in the morning. Hey! I don't make these things up!



A *relation* in $A \times B$ is any subset of $A \times B$. For example, in $\mathbb{R} \times \mathbb{R}$ the subset $L = \{(a,b) \mid a \in \mathbb{R} \text{ and } b \in \mathbb{R} \text{ and } a < b\}$ shows how we can represent the familiar relation "less than" in terms of a Cartesian product.

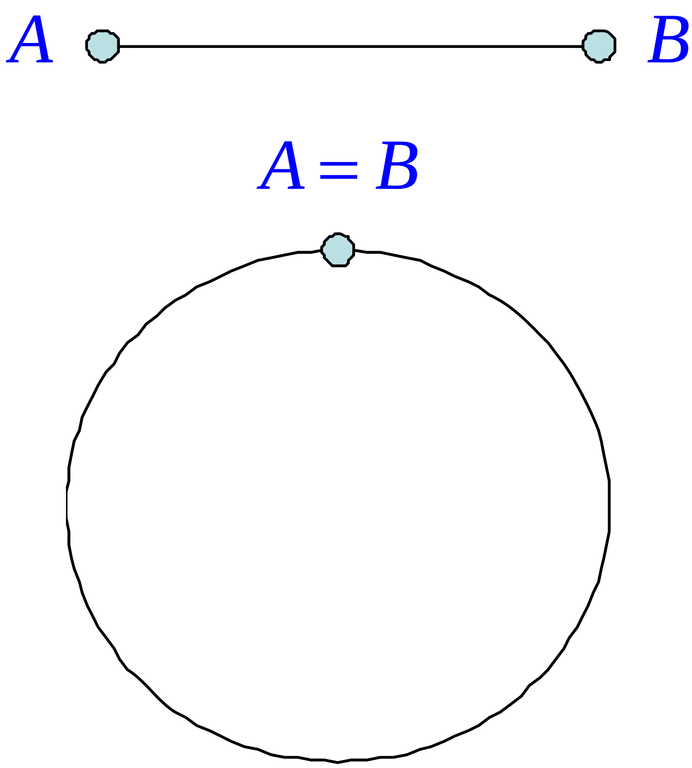
A *function* in $A \times B$ is a relation F such that if (x, y_1) and (x, y_2) belong to F, then $y_1 = y_2$. This is probably a very fancy way of saying something that you already know. Namely, that a function can't take a single element from its domain and pair it with more than one element in its range.

The equal sign was invented by the Welsh mathematician Robert Recorde in 1557, and since it consists of two parallel lines of equal length $(=)$, Recorde felt that there was no symbol better suited for denoting equality. Later mathematicians noted that equality has certain properties that we now call the *reflexive property* $(a = a)$, the *symmetric property* $(\text{If } a = b, \text{ then } b = a)$, and the *transitive property* $(\text{If } a = b \text{ and } b = c, \text{ then } a = c)$, and from these properties they abstracted to define an *equivalence relation*.

If A is a set and E is a relation in $A \times A$, then E is called an *equivalence relation* if the following conditions are met:

1. $\forall x \in A,\ (x,x) \in E$. (reflexive)
2. $\forall x, y \in A,$ if $(x,y) \in E,$ then $(y,x) \in E$. (symmetric)
3. $\forall x, y, z \in A,$ if $(x,y) \in E$ and $(y,z) \in E,$ then $(x,z) \in E$. (transitive)

If we are given for some set A an element a and an equivalence relation E in $A \times A$, then the set of all elements of A that are equivalent to a is called the *equivalence class of a*. I won't go into detail here, but suffice it to say that the concept of an equivalence class is one of the deepest and furthest reaching in all of mathematics. We can literally change our reality through what things we see as equivalent with one another. For example, we don't see *even numbers* until we see integers that are divisible by 2 as being "equivalent." Similarly, we don't see the forest until we see what makes one tree equivalent to another, and by making the two ends of a string equivalent to one another, we can turn a line segment into a circle.



If G is a set and if we have a function that goes from $G \times G \to G$, then we call this type of function a *binary operation*. When we have such a function, we don't normally use the $f(a,b)$ type of notation. Instead, we use maybe a symbol such as * for this operation and write something like $a*b$. For example, we can interpret "+" as a binary operation from $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$, and we commonly write things as $2+3=5$ rather than $f(2,3)=5$.

Using our definition of a binary operation and just a few of the algebraic properties for the real numbers, we can now define the concept of a *group* which is one of the most important algebraic structures of higher mathematics. Because there are so many things in the world of mathematics that are groups, a single theorem about groups applies to many different situations.

Let G be a non-empty set and let * be a binary operation defined on $G \times G$. Then G is a *group* if the following axioms are satisfied.

1. For every $a,b \in G$, $a*b \in G$. (closure)
2. For every $a,b,c \in G$, $a*(b*c)=(a*b)*c$. (associativity)

3. There exists an element $e \in G$ such that for every $a \in G$, $e * a = a = a * e$. (identity)
4. For every $a \in G$ there exists $a^{-1} \in G$ such that $a * a^{-1} = e = a^{-1} * a$. (inverses)

If the following additional property holds, then we call G a *commutative* or *abelian group*.

5. For every $a, b \in G$, $a * b = b * a$. (commutativity)

Again, the concept of a group is far reaching, and as you'll see in the exercises, groups are also associated with permutations and symmetry.

1. If $A = \{1,2,3\}$ and $B = \{4,5\}$, find $A \times B$, $B \times A$, and $B \times B$.

2. *Seven Brides for Seven Brothers*
   Let B be the set of brothers in this popular movie musical, and let
   $E = \{(a,b) \mid a, b \in B \text{ and } a \& b \text{ are brothers}\}$. Is E a relation in $B \times B$? If so, then is E an equivalence relation? Why or why not?

3. Read the article in the Wikipedia on René Descartes. Everyone should know something about Descartes.