Lesson 14

EQUIVALENCE RELATIONS

One of the simplest and yet most far reaching concepts in mathematics is that of an equivalence relation and an equivalence class. The notion is based upon the basic properties of equality that we call reflexive, symmetric, and transitive. These properties are illustrated below:

For numbers *a*, *b*, and *c*, the relationship of equality, denoted by "=," obeys the following properties.

1. $a = a$. (reflexive property)
2. If $a = b$, then $b = a$. (symmetric property)
3. If $a = b$ and $b = c$, then $a = c$. (transitive property)

We generalize from these properties of equality in order to define an <u>equivalence relation</u> which we will denote by "$\equiv$." (NOTE: A tilde, "~," is also frequently used in mathematics to denote equivalence between two objects.) Basically, we'll say that any relationship between any set of objects is an equivalence relation if it satisfies those three basic properties of being reflexive, symmetric, and transitive. And we should note that in this extension, our objects do not have to be numbers and the relationship can be however we define it. For example, suppose that our objects are people, and we will say that person *A* is related to person *B*, $A \equiv B$ (read as "*A* is equivalent to *B*"), if and only if *A* and *B* have the same last name with the same spelling. Well, a person has the same last name as themselves, so the relationship is clearly reflexive. Plus, if person *A* has the same last name as person *B*, then, likewise, person *B* has the same last name as person *A*, and so the relationship is also symmetric. And finally, if *A* has the same last name as *B* and *B* has the same last name as *C*, then clearly, *A* has the same as *C*, and, thus, our relationship is transitive and, hence, an equivalence relation.

One of the consequences of an equivalence relation is that it divides up our set into a series of non-overlapping subsets whose union is the original set. For example, when we say that two people will be considered equivalent if they have the same last name, then that divides up the entire population into several non-intersecting subsets. One subset will contain everyone named "Smith," another will contain everyone named "Jones," and so on. And occasionally, there may even be a rare name like "Mxyzptlk" that only a single person has. Either way, these non-overlapping subsets whose union is our entire population are called <u>equivalence classes</u>.

Basically, any partition of a set into a series of non-intersecting subsets whose union is the original universal set automatically defines an equivalence relation. For example, let's look at the set of elements in D_3, the dihedral group of degree 3, and we'll partition that set in a couple of ways. Our first partition will look like this.

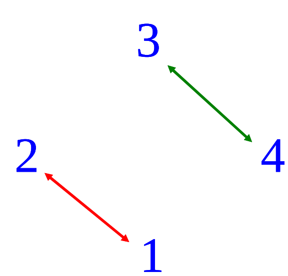
$(\ )$ $(1,2)$	$(1,2,3)$ $(1,3)$	$(1,3,2)$ $(2,3)$

And now, we can define two elements to be equivalent if they belong to the same subset in our partition. Thus, in the partition above we have that $(\ ) \equiv (1,2)$, $(1,2,3) \equiv (1,3)$, and $(1,3,2) \equiv (2,3)$. However, if we construct a different partition, then membership in the same subset will define a different equivalence relation.

$(\ )$ $(1,2,3)$ $(1,3,2)$		
$(1,2)$ $(1,3)$ $(2,3)$		

This partition defines an equivalence relation where $(\ ) \equiv (1,2,3) \equiv (1,3,2)$ and $(1,2) \equiv (1,3) \equiv (2,3)$. The bottom line, though, is that every equivalence relation defines a partition of our set into disjoint equivalence classes, and every partition of our set into non-intersecting subsets whose union is the original set, in turn, defines an equivalence relation.

We've already seen a few things that define equivalence relations for us. For example, consider the following generator diagram that we've seen before.



The group generated by these actions is the Klein 4-group with elements $G = \{(\ ),(1,2),(3,4),(1,2)(3,4)\}$. When we look at the orbits on $X = \{1,2,3,4\}$ generated by these permutations, we find just two, $\{1,2\}$ and $\{3,4\}$. Notice that $\{1,2\} \cap \{3,4\} = \varnothing$ and $\{1,2\} \cup \{3,4\} = X$. Basically, this means that saying "*a* is equivalent to *b* if and only if *a* and *b* belong to the same orbit" defines an equivalence relation on our set *X*.

Similarly, we can define an equivalence relation on *G* based upon whether a permutation fixes 1 (doesn't move 1 to another number) or doesn't fix 1 (does move 1 to another number). This relation will partition *G* into the following two non-intersecting sets of permutations, $\{(\ ),(3,4)\}$ and $\{(1,2),(1,2)(3,4)\}$. The permutations in the first set fix 1 (and form a stabilizer subgroup), and the permutations in the second set move 1. Furthermore, since $\{(\ ),(3,4)\}\cap\{(1,2),(1,2)(3,4)\}=\varnothing$ and $\{(\ ),(3,4)\}\cup\{(1,2),(1,2)(3,4)\}=G$, this partition defines an equivalence relation.

In mathematics, a very important type of equivalence relation is conjugacy, or, in other words, if *a* and *b* are elements of a group *G*, then we'll say that $a\equiv b$ if *a* is conjugate to *b*. Recall that that means that there is an element *c* in *G* such that $c^{-1}ac=b$. For example, the conjugacy classes of the dihedral group D_3 partition the set $X=\{1,2,3\}$ into three disjoint (non-intersecting) subsets, $\{(\ )\}$ and $\{(1,2),(1,3),(2,3)\}$ and $\{(1,2,3),(1,3,2)\}$. These non-overlapping subsets are our conjugacy classes, and notice that their union is D_3. By the way, the group D_3 in not abelian, but notice that if our group were abelian, then we would always have $c^{-1}ac=c^{-1}ca=ea=a$. In other words, in an abelian (commutative) group, each element represents its own conjugacy class.

Equivalence relations are not only important and ubiquitous throughout all of higher mathematics, they also determine how we view the world. For example, take everything you know and classify it as "good" or "bad." Many of us will wind up with very similar lists for what is good and what is bad, but when it comes to controversial topics like gun control, abortion, and immigration policies, people often have very different opinions on what's good and what's bad. Nonetheless, the manner in which we classify elements of our reality as either good or bad results in a particular worldview that may be very different from that of some of our peers. Our brains routinely use equivalence classes to help create our unique perception of reality.