DOC BENTON'S JOURNAL

(a work in progress)

Theorem: If *G* is a *group* and if *H* and *N* are *subgroups* of *G* with *N normal* in *G,* then *HN* is a *subgroup* of *G*.

Proof: To show that *HN* is a *subgroup* of *G*, we need to verify *closure* and existence of *inverses*. Thus, let $h_1, h_2 \in H$ and let $n_1, n_2 \in N$. Then $h_1n_1, h_2n_2 \in HN$. Now consider $h_1n_1 \cdot h_2n_2 = h_1(n_1h_2)n_2$. Since $N \lhd G$, there exists $n_3 \in N$ such that $n_1h_2 = h_2n_3$. Hence, $h_1n_1 \cdot h_2n_2 = h_1(n_1h_2)n_2 = h_1(h_2n_3)n_2 = h_1h_2 \cdot n_3n_2 \in HN$, and thus, closure is satisfied.

Now let $h \in H$ and let $n \in N$. Then $(hn)^{-1} = n^{-1}h^{-1}$. However, again, since $N \lhd G$, there exists $n_4 \in N$ such that $n^{-1}h^{-1} = h^{-1}n_4 \in HN$, and thus, *inverses* exist. Therefore, $HN \le G$.

□

Theorem: If *G* is a *group* and if *H* and *N* are both *normal subgroups* of *G,* then *HN* is a *normal subgroup* of *G*.

Proof: By the previous proof, we know that *HN* is a *subgroup* of *G*. Now let $g \in G$, and consider $g \cdot HN \cdot g^{-1}$. Clearly, $g \cdot HN \cdot g^{-1} = g \cdot H \cdot e \cdot N \cdot g^{-1} = g \cdot H \cdot g^{-1} g \cdot N \cdot g^{-1} = (g \cdot H \cdot g^{-1})(g \cdot N \cdot g^{-1}) = HN$ since both *H* and *N* are *normal* in *G*. Therefore, $HN \triangleleft G$.

$\square$

Theorem: If *G* is a *group* and if *H* and *N* are both *normal subgroups* of *G,* then $H \bigcap K$ is a *normal subgroup* of *G*.

Proof: Let $g \in G$ and let $x \in H \bigcap K$, and consider gxg^{-1}. Since $x \in H$ and $H \triangleleft G$, it follows that $gxg^{-1} \in H$. But on the other hand, since $x \in K$ and $K \triangleleft G$, it follows that $gxg^{-1} \in K$. Therefore, $gxg^{-1} \in H \bigcap K$, and $H \bigcap K \triangleleft G$.

□

<u>Theorem:</u> Let *G* be a *group* with *subgroups* *H* and *K* that are not necessarily *normal subgroups*. Then $\left|\frac{H}{H \cap K}\right| = \left|\frac{HK}{K}\right|$.

<u>Proof:</u> Since $H, K \leq G$ it follows from previous proofs that $H \cap K \leq G$,and hence, we can consider the *left cosets* of $H \cap K$ in *G*, and since clearly $H \cap K \subseteq H$, wee can also consider the *left cosets* of $H \cap K$ in *H*. Furthermore, the number of such *cosets* is represented by $\left|\frac{H}{H \cap K}\right|$. Additionally, even though we don't know if *HK* is a *subgroup* of *G* (since we don't know if *H* or *K* is *normal* in *G*), it is still obvious that $K \subseteq HK$, and hence, we can denote the number of *left cosets* of *K* in *HK* by $\left|\frac{HK}{K}\right|$.

To show that $\left|\frac{H}{H \cap K}\right| = \left|\frac{HK}{K}\right|$, it will suffice to find a function $f : \frac{H}{H \cap K} \to \frac{HK}{K}$ that is a *bijection*. Hence, if $h(H \cap K) \in \frac{H}{H \cap K}$, then let $f(h(H \cap K)) = hK \in \frac{HK}{K}$. We now need to show that *f* is *one-to-one* and *onto*. To show that it is *onto*, let $hk \in HK$ where $h \in H$ and $k \in K$. Then $hk(K) = hK = f(h(H \cap K))$, and hence, *f* is *onto*.

To show that *f* is *one-to-one*, let $h_1(H \cap K), h_2(H \cap K) \in \frac{H}{H \cap K}$ such that $f(h_1(H \cap K)) = f(h_2(H \cap K))$. Then $h_1K = h_2K \Rightarrow h_1 = h_2k$ for some $k \in K$. Hence, $h_2^{-1}h_1 = k \Rightarrow h_2^{-1}h_1 \in H$ & $h_2^{-1}h_1 \in K \Rightarrow h_2^{-1}h_1 \in H \cap K$. But his also implies that $h_2^{-1}h_1(H \cap K) = H \cap K$ which, in turn, implies that $h_1(H \cap K) = h_2(H \cap K)$. Consequently, *f* is also *one-to-one*, and therefore *f* is a *bijection*, and thus, $\left|\frac{H}{H \cap K}\right| = \left|\frac{HK}{K}\right|$.

□

<u>Corollary:</u> If *G* is a *group* and $H, K \leq G$, then $|HK| = \dfrac{|H||K|}{|H \cap K|}$.

<u>Proof:</u> Our theorem shows us that $\left|\dfrac{H}{H \cap K}\right| = \left|\dfrac{HK}{K}\right|$, and Lagrange's Theorem tells us that $\left|\dfrac{H}{H \cap K}\right| = \dfrac{|H|}{|H \cap K|}$ and $\left|\dfrac{HK}{K}\right| = \dfrac{|HK|}{|K|}$. From this it easily follows that $\dfrac{|H||K|}{|H \cap K|} = |HK|$.

□

Theorem (The Firth Isomorphism Theorem): If $A, B \triangleleft G$ and $f: G \to G/B$ is the *natural homomorphism*, then $\frac{G}{AB} \cong \frac{G/B}{f(A)}$.

Proof: This really follows from the 2nd and 3rd *Isomorphism Theorems*. In particular, notice that as sets $\frac{AB}{B}$ and $\frac{A}{A \bigcap B}$ are each composed of a set which is factored out (which for convenience we will refer to as the *identity element* in each respective *quotient*) and the same elements of *A* that don't get mapped to the *identity* in each respective *quotient structure*. More specifically, as groups we have by the 2nd *Isomorphism Theorem* that $\frac{A}{A \bigcap B} \cong \frac{AB}{B}$ and, furthermore, $f(A) = \frac{AB}{B}$. Thus, by the 3rd *Isomorphism Theorem*, we have that $\frac{G/B}{f(A)} = \frac{G/B}{(AB)/B} \cong \frac{G}{AB}$.

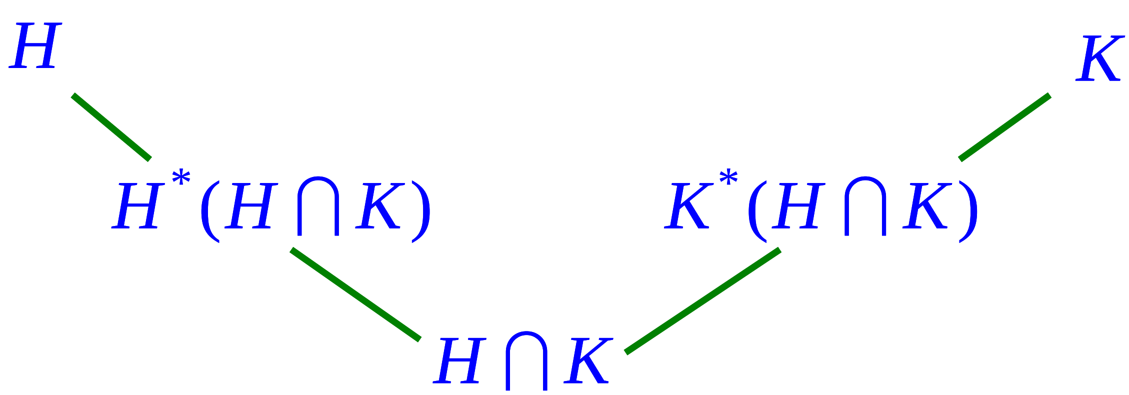
□

NOTE: The essence of this result is that we can think of arriving at $\frac{G}{AB}$ by first factoring out *B* and then factoring of what left of the elements of *A* in $\frac{G}{B}$. Additionally, we can weaken the conditions by requiring only that *B* and *AB* be *normal subgroups* of *G*.

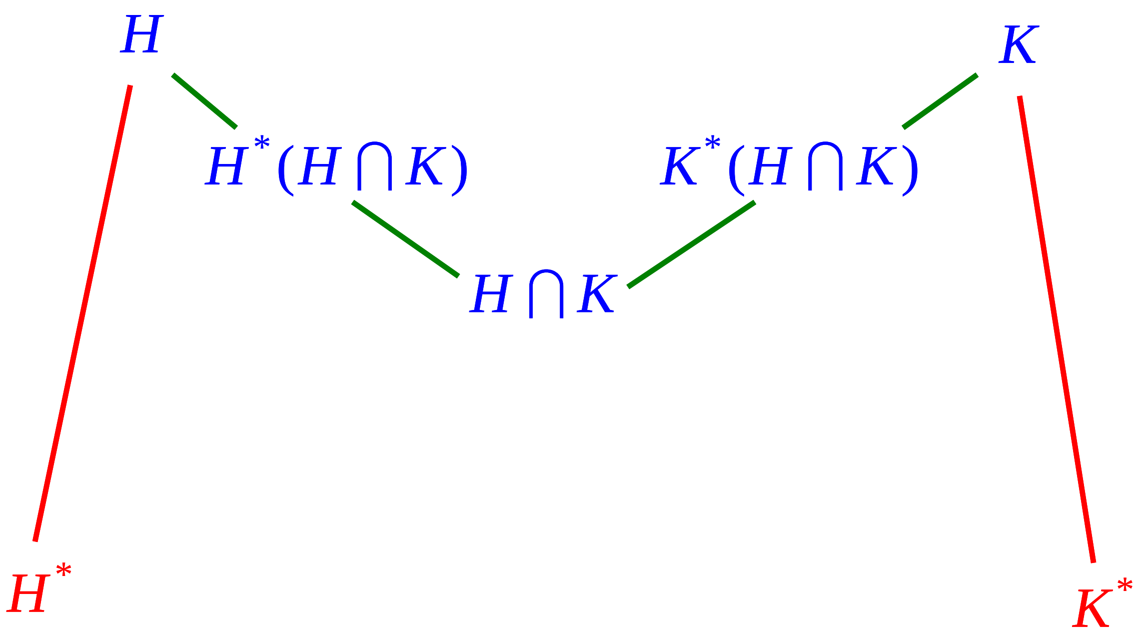
<u>Theorem (Zassenhaus Butterfly Lemma):</u> Let *H* and K be *subgroups* of a *group* *G*, and suppose *H** and *K** are *subgroups* of *G* such that $H^* \lhd H$ and $K^* \lhd K$. Then,

1. $H^*(H \cap K^*) \lhd H^*(H \cap K)$,
2. $K^*(H^* \cap K) \lhd K^*(H \cap K)$, and
3. $\dfrac{H^*(H \cap K)}{H^*(H \cap K^*)} \cong \dfrac{K^*(H \cap K)}{K^*(H^* \cap K)} \cong \dfrac{H \cap K}{(H^* \cap K)(H \cap K^*)}$.

<u>Proof:</u> This theorem is often called the *Butterfly Lemma* because of a diagram we'll construct for a lattice of *subgroups* that is reminiscent of a butterfly. Thus, let's begin by identifying some *subgroups* to include in our diagram. First, we have that since *H* and *K* are *subgroups* of *G*, so is their intersection $H \cap K$ also a *subgroup* of *G*. Note, too, that $H \cap K \leq H$ and $H \cap K \leq K$. Also, the product $H^*(H \cap K)$ is a *subgroup* of *H* since it is the product of a *normal subgroup* of *H* with a *subgroup* of *H*, and likewise, $K^*(H \cap K) \leq K$ since it is the product of a *normal subgroup* of *K* with a *subgroup* of *K*. These relationships are shown visually by the diagram below.

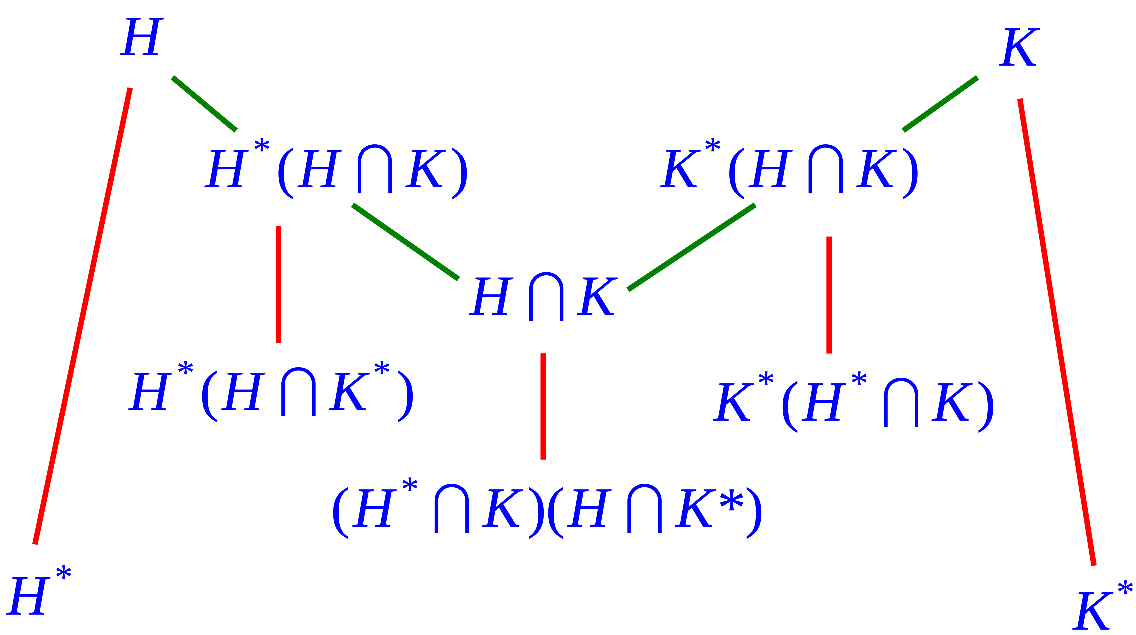


Next, since H^* is a *normal subgroup* of *H* and K^* is a *normal subgroup* of *K*, we'll add these *subgroups* to our diagram, and since they are *normal subgroups*, we'll denote relationship this in the lattice by red lines.

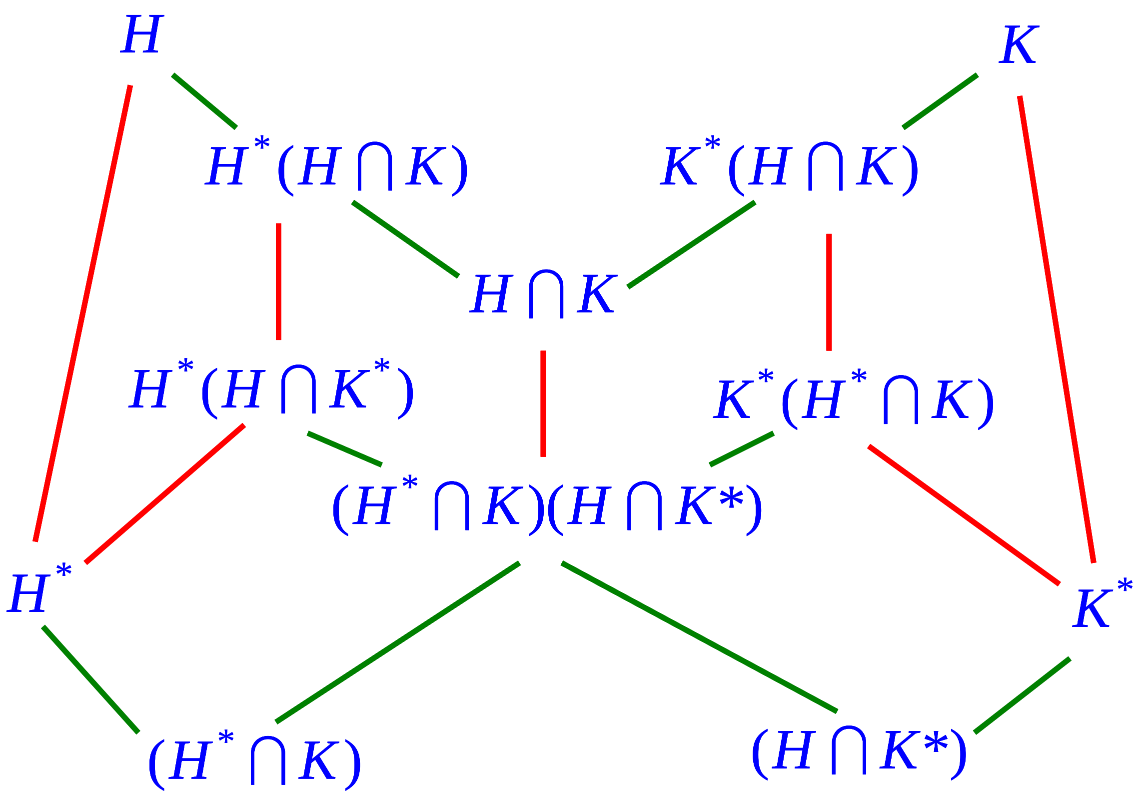


Next, we have that $H^*(H \cap K^*) \lhd H^*(H \cap K)$ and $K^*(H^* \cap K) \lhd K^*(H \cap K)$.. Here's how we can start to verify this. We will begin by showing that $H \cap K^* \lhd H \cap K$. Thus, suppose $x \in H \cap K^*$ and $g \in H \cap K$. Then $x \in K^*$ and $g \in K$, and since $K^* \lhd K$, it follows that $gxg^{-1} \in K^*$. But on the other hand, it also follows from $x \in H \cap K^*$ and $g \in H \cap K$ that $x \in H$ and $g \in H$, and thus, $gxg^{-1} \in H$ as well. Hence, we can now conclude that $gxg^{-1} \in H \cap K^*$ and therefore, $H \cap K^* \lhd H \cap K$. From this it follows that at the very least, we have that $H \cap K^* \leq H^*(H \cap K)$, Furthermore, since every element of $H^*(H \cap K)$ is also an element of *H* and since $H^* \lhd H$, it follows that $H^* \lhd H^*(H \cap K)$. Also, using similar arguments we can conclude that $H^* \cap K \lhd H \cap K$ and $H^* \cap K \leq H^*(H \cap K)$ and $K^* \lhd K^*(H \cap K)$. And now using the result of our recent theorem that the product of a *normal subgroups* and a *subgroup* is a *subgroup*, we can conclude that

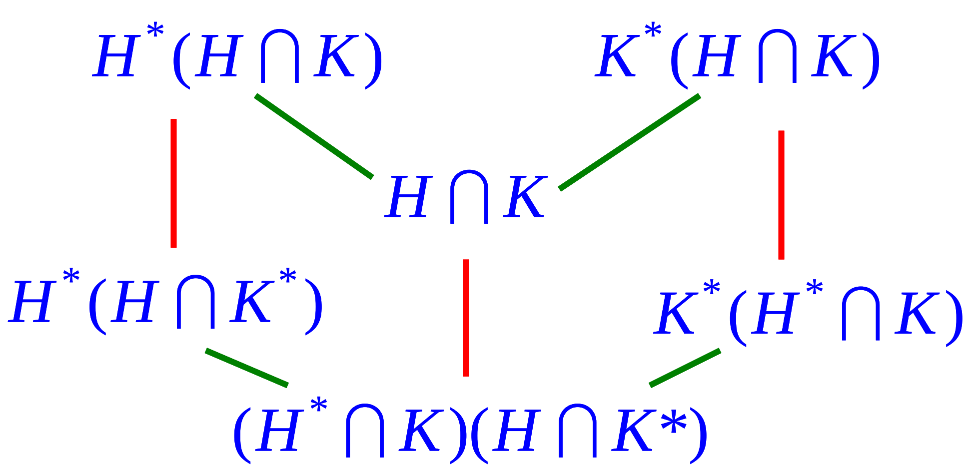
$H^*(H \cap K^*) \leq H^*(H \cap K)$. It will actually turn out that $H^*(H \cap K^*)$ is a *normal subgroup* of $H^*(H \cap K)$ and $K^*(H^* \cap K)$ is a *normal subgroup* of $K^*(H \cap K)$, but we will prove that later by showing that $H^*(H \cap K^*)$ and $K^*(H^* \cap K)$ are the *Kernels* of *homomorphisms*. Next, we have that $(H^* \cap K)(H \cap K^*)$ is a product of two *normal subgroups* of $H \cap K$, and therefore it follows from one of our previous theorems that $(H^* \cap K)(H \cap K^*) \lhd H \cap K$. Below we illustrate all of these relationships by adding on a bit to our previous diagram.



And finally, it should be fairly obvious that we have the following *subgroup* relationships - $H^* \lhd H^*(H \cap K^*)$, $K^* \lhd K^*(H^* \cap K)$, $(H^* \cap K) \leq H^*$, $(H \cap K^*) \leq K^*$, $(H^* \cap K)(H \cap K^*) \leq H^*(H \cap K^*)$, $(H^* \cap K)(H \cap K^*) \leq K^*(H^* \cap K)$. $(H^* \cap K) \leq (H^* \cap K)(H \cap K^*)$, and $(H \cap K^*) \leq (H^* \cap K)(H \cap K^*)$. These relationships are illustrated below in our final butterfly diagram.



Of all these relationships, however, there are only going to be three that are important to us. In particular, $H^*(H \cap K^*) \triangleleft H^*(H \cap K)$, $K^*(H^* \cap K) \triangleleft K^*(H \cap K)$, and $(H^* \cap K)(H \cap K^*) \triangleleft (H \cap K)$. This is illustrated by the diagram below.



The *Zassenhaus Butterfly Lemma* now claims that the following three *quotient groups* are *isomorphic* - $\frac{H^*(H\cap K)}{H^*(H\cap K^*)}\cong\frac{K^*(H\cap K)}{K^*(H^*\cap K)}\cong\frac{H\cap K}{(H^*\cap K)(H\cap K^*)}$. Notice, however, that due to symmetry we already have that $\frac{H^*(H\cap K)}{H^*(H\cap K^*)}\cong\frac{K^*(H\cap K)}{K^*(H^*\cap K)}$. All one needs to do is to re-label *H* as *K* and *K* as *H* in order to obtain this result. Thus, the crux of the proof is going to be to show that $\frac{H^*(H\cap K)}{H^*(H\cap K^*)}\cong\frac{H\cap K}{(H^*\cap K)(H\cap K^*)}$. And to do this, we're going to take a different route than most proofs. Specifically, we're going to be factoring out one *normal subgroup* at a time in order to show:

1. $\frac{H^*(H\cap K)}{H^*}\cong\frac{H\cap K}{H^*\cap K}$,
2. $\frac{(H^*\cap K)(H\cap K^*)}{H^*\cap K}\lhd\frac{H\cap K}{H^*\cap K}$,
3. $\frac{(H^*\cap K)(H\cap K^*)}{H^*\cap K}\cong\frac{H^*(H\cap K^*)}{H^*}\lhd\frac{H^*(H\cap K)}{H^*}$, and
4. $\dfrac{\frac{H^*(H\cap K)}{H^*}}{\frac{H^*(H\cap K^*)}{H^*}}\cong\dfrac{\frac{H\cap K}{H^*\cap K}}{\frac{(H^*\cap K)(H\cap K^*)}{H^*\cap K}}\Rightarrow\frac{H^*(H\cap K)}{H^*(H\cap K^*)}\cong\frac{H\cap K}{(H^*\cap K)(H\cap K^*)}$.

Theorem (Zassenhaus Butterfly Lemma): Let *H* and K be *subgroups* of a *group* *G*, and suppose *H** and *K** are *subgroups* of *G* such that $H^* \lhd H$ and $K^* \lhd K$. Then,

1. $H^*(H \cap K^*) \lhd H^*(H \cap K)$,
2. $K^*(H^* \cap K) \lhd K^*(H \cap K)$, and
3. $\dfrac{H^*(H \cap K)}{H^*(H \cap K^*)} \cong \dfrac{K^*(H \cap K)}{K^*(H^* \cap K)} \cong \dfrac{H \cap K}{(H^* \cap K)(H \cap K^*)}$.

Proof: To prove this lemma, we'll make use of both the *Second Isomorphism Theorem* and what I've called the *Fifth Isomorphism Theorem.* Thus, recall that the *Second Isomorphism Theorem* tell us, given appropriate normality of the groups involved, that $\dfrac{A}{A \cap B} \cong \dfrac{AB}{B}$, and likewise the *Fifth Isomorphism Theorem* tells us that $\dfrac{G}{AB} \cong \dfrac{G/B}{f(A)} \cong \dfrac{G/B}{AB/B}$ where $f : G \to \dfrac{G}{B}$ is the natural homomorphism and we again assume normality where appropriate. Thus, let's begin by looking more closely at $H^*(H \cap K^*)$. We have that $H^* \lhd H^*(H \cap K)$, and by the *Second Isomorphism Theorem*, it follows that $\dfrac{H^*(H \cap K)}{H^*} \cong \dfrac{H \cap K}{H^* \cap (H \cap K)} = \dfrac{H \cap K}{H^* \cap K}$, a *quotient group* of $H \cap K$.

Now let's examine $H \cap K$ in more detail. First, we know that $H \cap K^* \lhd H \cap K$. Now think about what happens if we map $H \cap K^*$ into $\dfrac{H \cap K}{H^* \cap K}$ using the *natural homomorphism* which I'll denote by $f : H \cap K \to \dfrac{H \cap K}{H^* \cap K}$. Basically, everything in $H^* \cap (H \cap K^*)$ gets mapped to the identity in $\dfrac{H \cap K}{H^* \cap K}$ and the rest gets mapped to non-identity elements. Also, by the *Second Isomorphism Theorem* we have that $\dfrac{H \cap K^*}{(H^* \cap K) \cap (H \cap K^*)} \cong \dfrac{(H^* \cap K)(H \cap K^*)}{H^* \cap K}$. This latter *quotient* shows us the *subgroup* of $\dfrac{H \cap K}{H^* \cap K}$ that corresponds to $H \cap K^* \lhd H \cap K$. In other words,

$$f(H \cap K^*) = \frac{(H^* \cap K)(H \cap K^*)}{H^* \cap K} \quad \text{and} \quad f^{-1}\left(\frac{(H^* \cap K)(H \cap K^*)}{H^* \cap K}\right) = (H^* \cap K)(H \cap K^*) \quad .$$

Furthermore, $(H^* \cap K)(H \cap K^*) \lhd H \cap K \Rightarrow \dfrac{(H^* \cap K)(H \cap K^*)}{H^* \cap K} \lhd \dfrac{H \cap K}{H^* \cap K}$.

Notice also that since $\dfrac{H \cap K}{H^* \cap K} \cong \dfrac{H^*(H \cap K)}{H^*}$, there is a *normal subgroup N* of $\dfrac{H^*(H \cap K)}{H^*}$ that corresponds to $\dfrac{(H^* \cap K)(H \cap K^*)}{H^* \cap K}$. We can discover the structure of this *normal subgroup* by again applying the *Second Isomorphism Theorem*. In particular,

$$N \cong \frac{(H^* \cap K)(H \cap K^*)}{H^* \cap K} \cong \frac{H \cap K^*}{(H^* \cap K) \cap (H \cap K^*)} = \frac{H \cap K^*}{H^* \cap (H \cap K^* \cap K)} = \frac{H \cap K^*}{H^* \cap (H \cap K^*)}$$

$$\cong \frac{H^*(H \cap K^*)}{H^*} \lhd \frac{H^*(H \cap K)}{H^*}.$$

Hence, $$\frac{\dfrac{H^*(H \cap K)}{H^*}}{\dfrac{H^*(H \cap K^*)}{H^*}} \cong \frac{\dfrac{H \cap K}{H^* \cap K}}{N} \cong \frac{\dfrac{H \cap K}{H^* \cap K}}{\dfrac{(H^* \cap K)(H \cap K^*)}{H^* \cap K}} \Rightarrow \frac{H^*(H \cap K)}{H^*(H \cap K^*)} \cong \frac{H \cap K}{(H^* \cap K)(H \cap K^*)} \quad .$$

This essentially proves (1) and the first part of (3) above, and to prove (2) and the rest of (3), we can essentially just repeat the same proof with the roles of *H* and *K* reversed. In other words, we just take advantage of the symmetry that exists between the formulas $H^*(H \cap K^*) \lhd H^*(H \cap K)$ and $K^*(H^* \cap K) \lhd K^*(H \cap K)$.

□

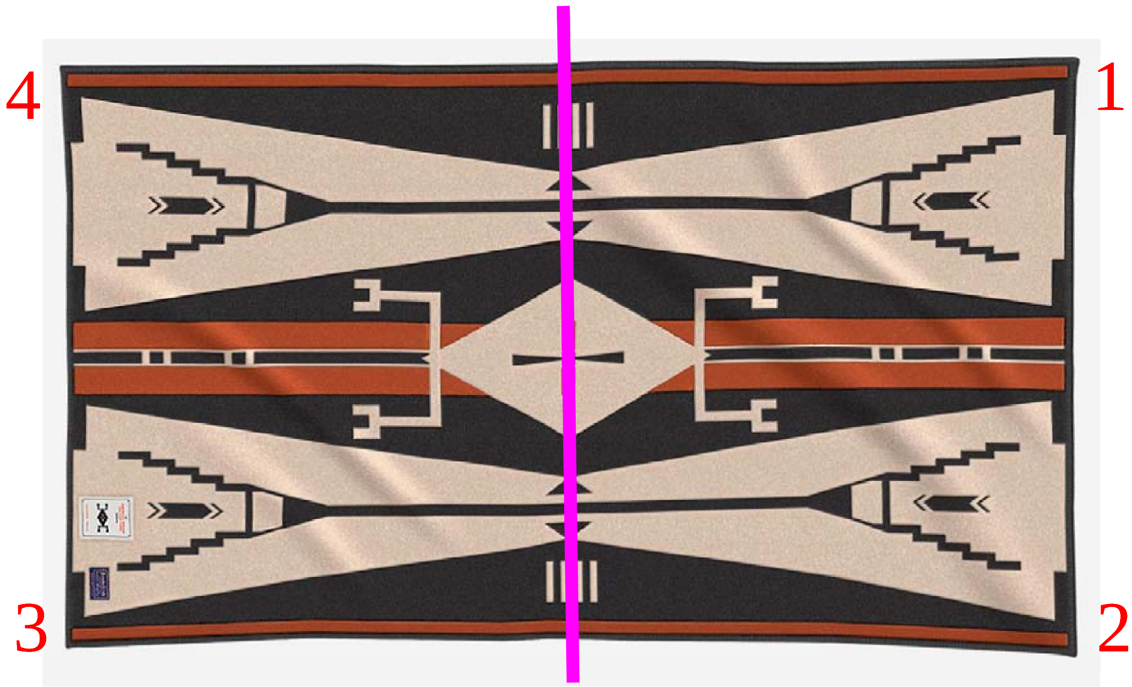
NATIVE AMERICAN DESIGN-1

Below is a fairly common Native American design, and our task will be to analyze the symmetry found within it.

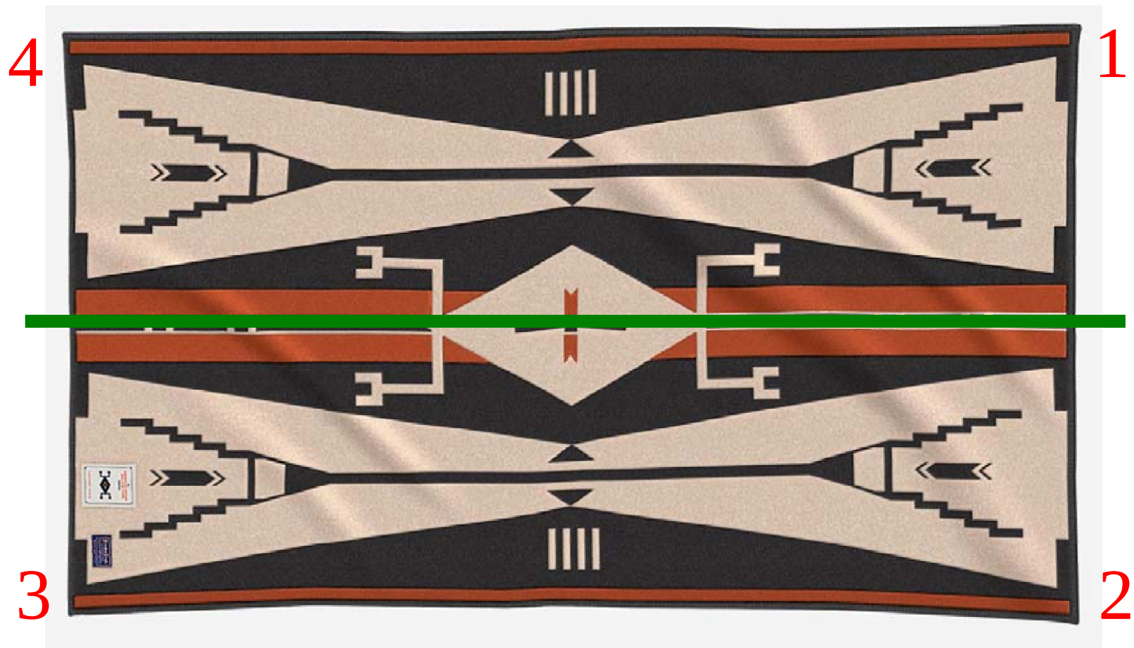
One thing that might assist us in associating a mathematical *group* to this design would be to first label each corner with a number.

If we look closely at the design, we might notice that there are mirror lines present. In particular, we can do a reflection across a vertical line through the center, and we can also do a reflection about a horizontal line through the center. Additionally, we can also rotate our design 180 degrees either clockwise or counterclockwise about the center. And lastly, each reflection or rotation will cause a permutation of the numbers that we have used to label the corners of our design.

If we flip our design about the vertical axis of reflection, then 1 & 4 will switch places as will 2 & 3. We can represent this change as the product of the following two cycles, (1, 4)(2, 3).



Similarly, if we flip our design about the horizontal axis of reflection, then 1 & 2 and 3 & 4 will change places. That change can be represented by this product of two cycles, (1, 2)(3, 4).

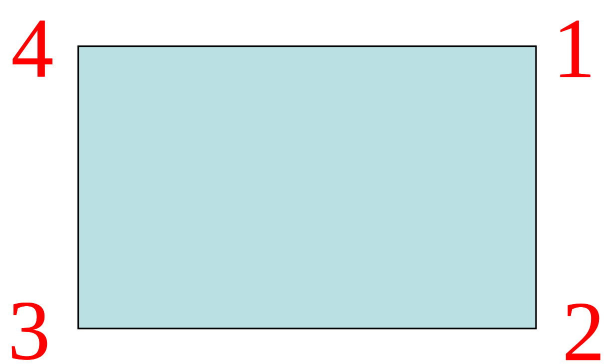


And finally, if we rotate our design 180 degrees clockwise about the center, then that will correspond to this permutation of the numbered corners, (1, 3)(2, 4).

For our next step, we can enter the above permutations into our GAP program (Groups, Programming, and Algorithms) and see what *group* they generate.

```
gap> a:=(1,4)(2,3);
(1,4)(2,3)

gap> b:=(1,2)(3,4);
(1,2)(3,4)

gap> c:=(1,3)(2,4);
(1,3)(2,4)

gap> g:=Group(a,b,c);
Group([ (1,4)(2,3), (1,2)(3,4), (1,3)(2,4) ])

gap> Size(g);
4

gap> IsCyclic(g);
false
gap>
```

From the output above we see that our cycles create a *group* of order 4 and that it is not a *cyclic group*. Since there are only two *groups* of order 4, the *cyclic group* of order 4 and the *Klein 4-group*, it follows that this *group* is the *Klein 4-group*. This is the same *group* that corresponds to the symmetry of a rectangle.

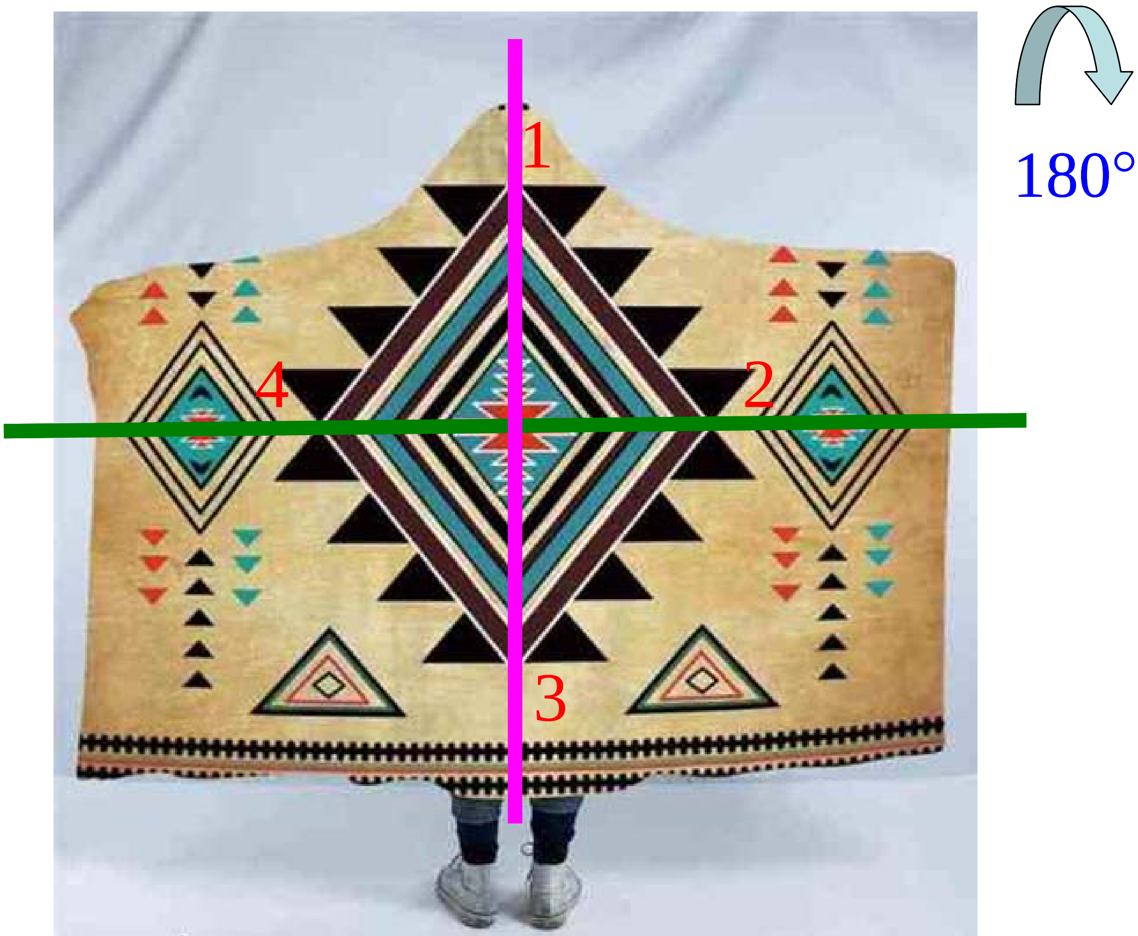


Additionally, notice that (1, 4)(2, 3)*(1, 2)(3, 4) = (1, 3)(2, 4). In other words, the rotation can be generated by a vertical axis flip followed by a horizontal axis flip (multiplying cycles from left to right), and thus, the enter *group* is generated just by the reflections.

NATIVE AMERICAN DESIGN-2

Below is another very common Native American design, and as before, we'll assign numbers to some of the points in order to help us analyze the symmetry.

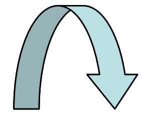
In examining the design, we can see both rotational symmetry of order 2 (through an angle of 180 degrees) as well as reflections about both a vertical axis and a horizontal axis.



If we do a reflection about the vertical axis, then that will correspond to the cycle (2, 4). The number 2 will move to where 4 currently is, and the number 4 will move to where 2 currently is. Everything else will stay the same.

If we do a reflection about the horizontal axis, then that will correspond to the cycle (1, 3). The number 1 will move to where 3 currently is, and the number 3 will move to where 1 currently is. Everything else will stay the same.

And finally, if we do a rotation clockwise through 180 degrees, then that will move 1 to 3 and 2 to 4, or in other words, create the cycle product (1, 3)(2, 4).

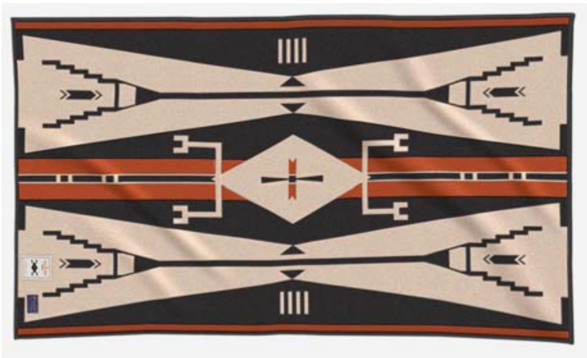


180°

If we enter these cycles into GAP and examine the *group* that they generate, then we get back the following.

```
gap> a:=(2,4);
(2,4)

gap> b:=(1,3);
(1,3)

gap> c:=(1,3)(2,4);
(1,3)(2,4)

gap> g:=Group(a,b,c);
Group([ (2,4), (1,3), (1,3)(2,4) ])

gap> Size(g);
4

gap> IsCyclic(g);
false
```

And now, as we can see, we have once again created a *group* of order 4 that is not cyclic, and hence, we are looking at another representation of the *Klein 4-group*. Hence, in these first two examples we have two different patterns, but they correspond to the same *symmetry group* which can be generated by two reflections, one vertical and the other horizontal.



NATIVE AMERICAN DESIGN-3

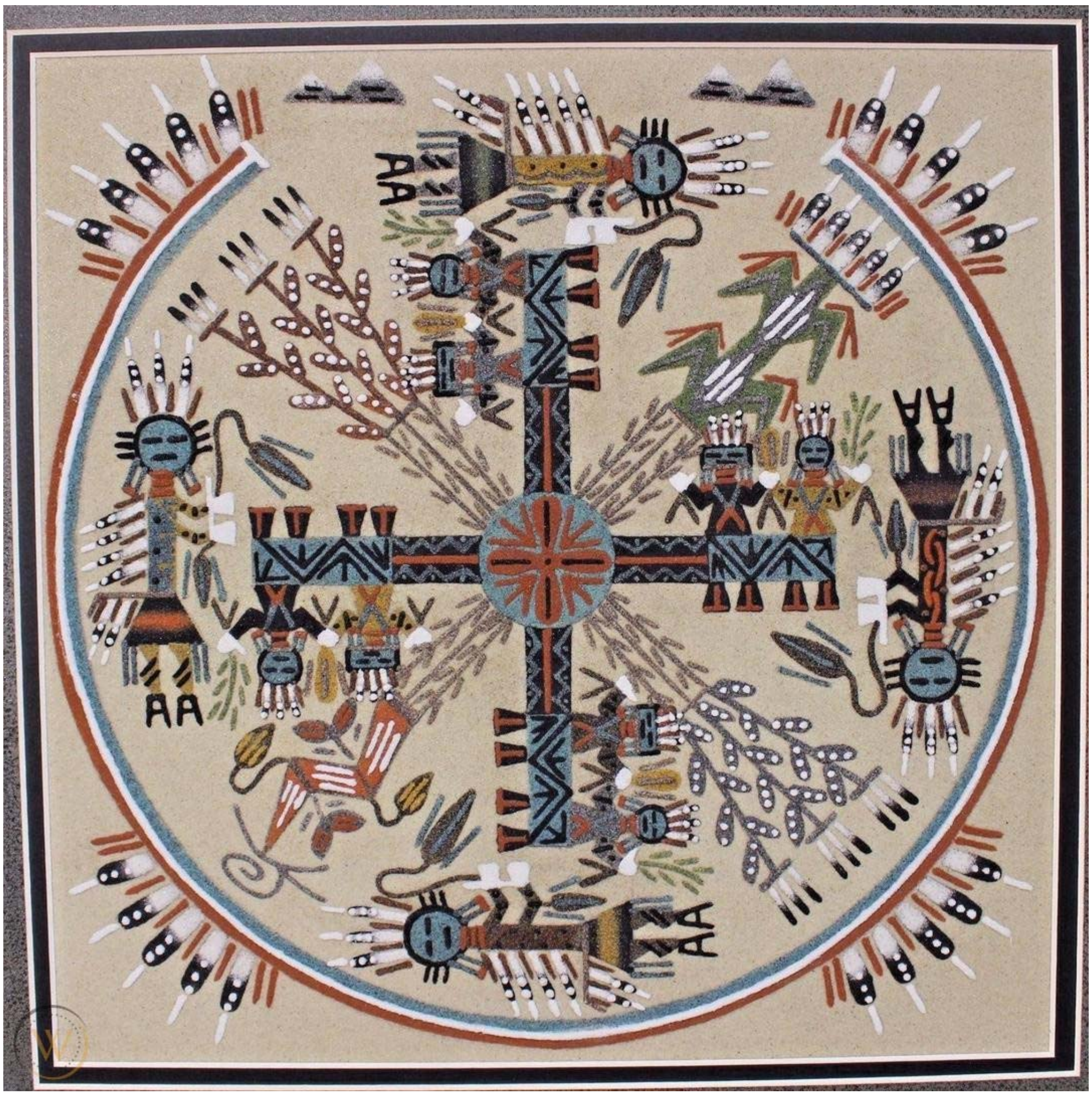
Below are two circular design by artists in the Pacific Northwest, and the most interesting thing about them may be their lack of symmetry. Furthermore, little or no symmetry seems to be common to many Native designs from this part of the country. In this design there is neither rotational symmetry nor mirror lines that the image can be reflected across. Instead, we essentially have a circle with a variety of images contained inside. By itself a circle has an infinite amount of symmetry. For example, a circle may be rotated through an infinite number of angles or reflected across an infinite number of lines that pass through its center. However, none of the images inside our circle take advantage of any of these opportunities for a more symmetrical design. Nonetheless, there is a definite style to this type of artwork that involves broad strokes of black lines and other colors, and this repetition of style within each design is itself a type of symmetry. Still, there is no definitive symmetry group that can be associated with these designs.

NATIVE AMERICAN DESIGN-4

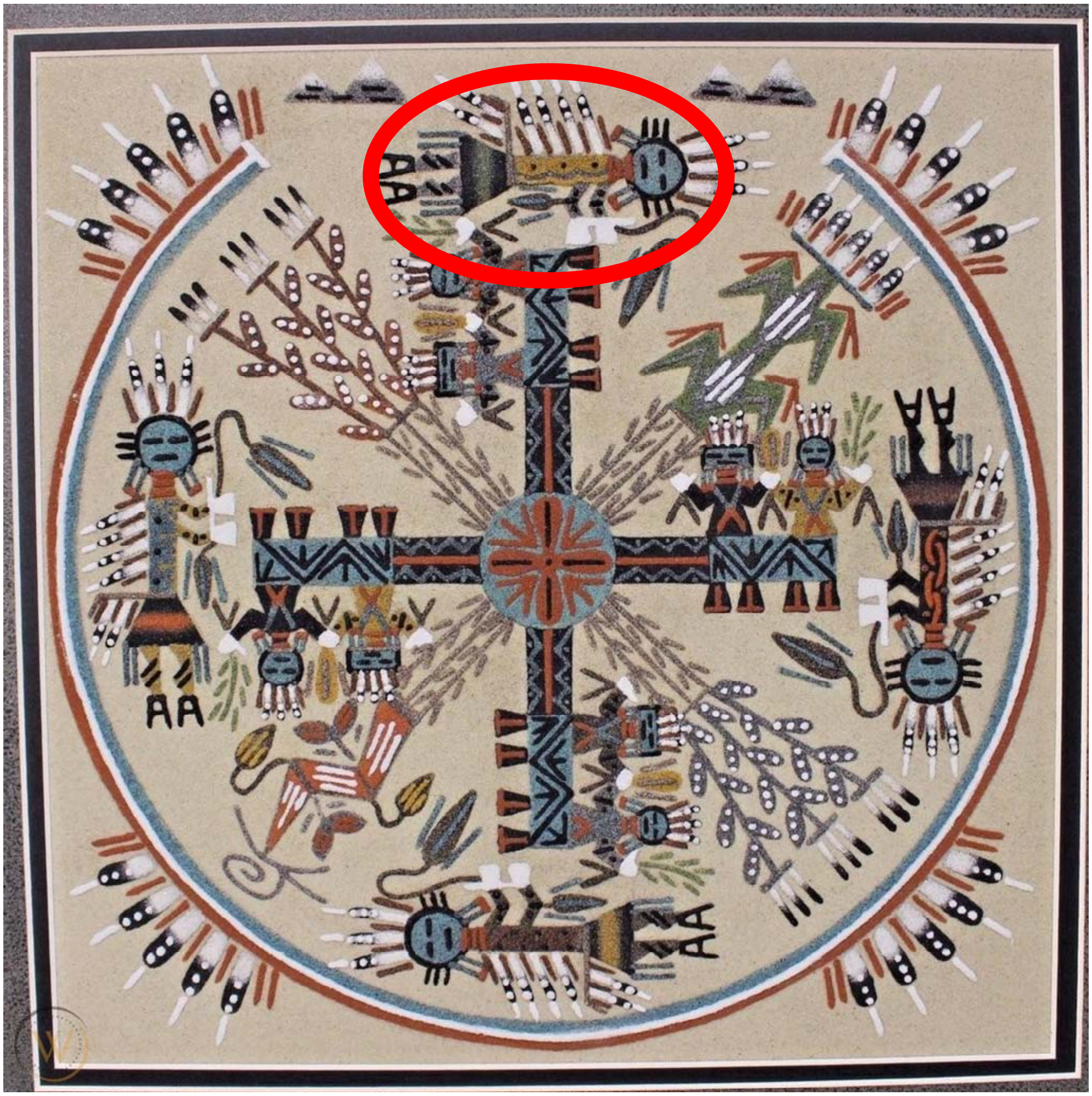
Here is an image of a totem pole such as is found among certain tribes in the American Northwest. In these totem pole images we generally find bilateral symmetry such as in the image below. However, that is generally about all that we find.

NATIVE AMERICAN DESIGN-5

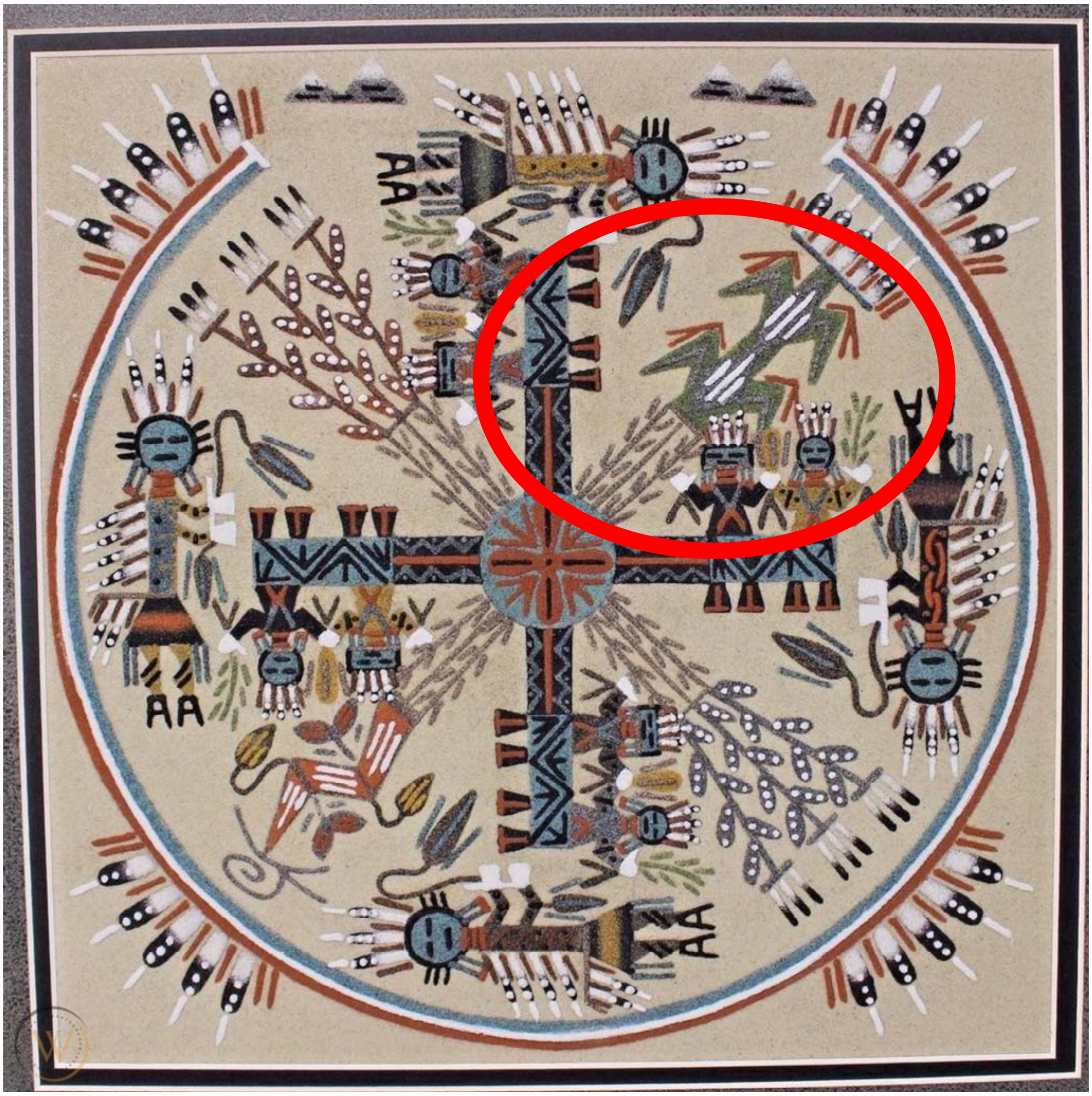
Below is a Navajo sandpainting, and at first glance it appears to have both rotational and mirror symmetry and a symmetry *group* corresponding the *dihedral group* D_4, or at least the *cyclic group* C_4. However, from a strict mathematical point of view the image below, surprisingly, has no symmetry! This is because there are items that break the apparent symmetry in the image.



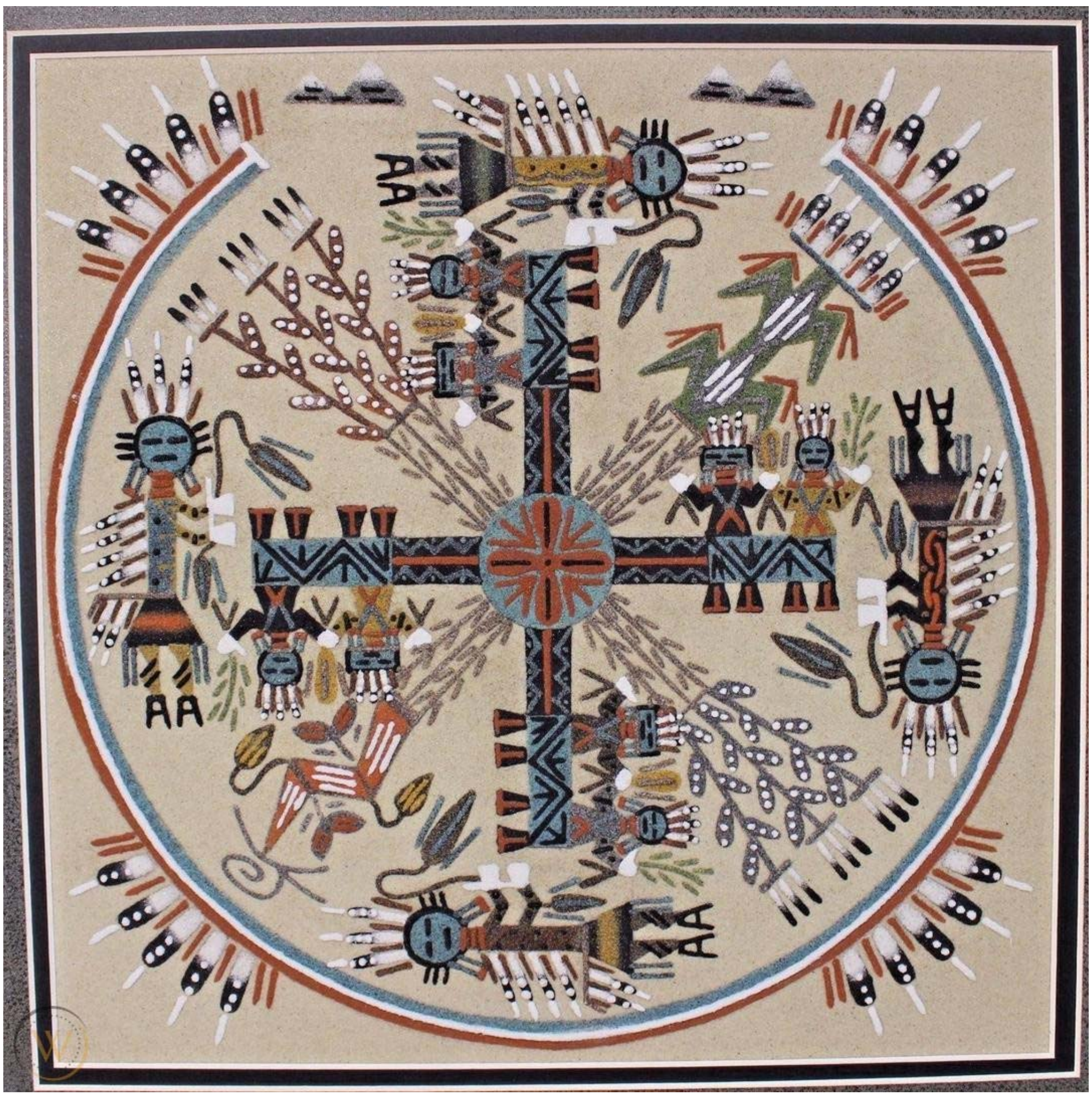
For example, mirror symmetry is broken by the fact that the figure circled in the sandpainting below sticks out to the right and not to the left. This results in any vertical, horizontal, or diagonal reflections looking different from the original image.



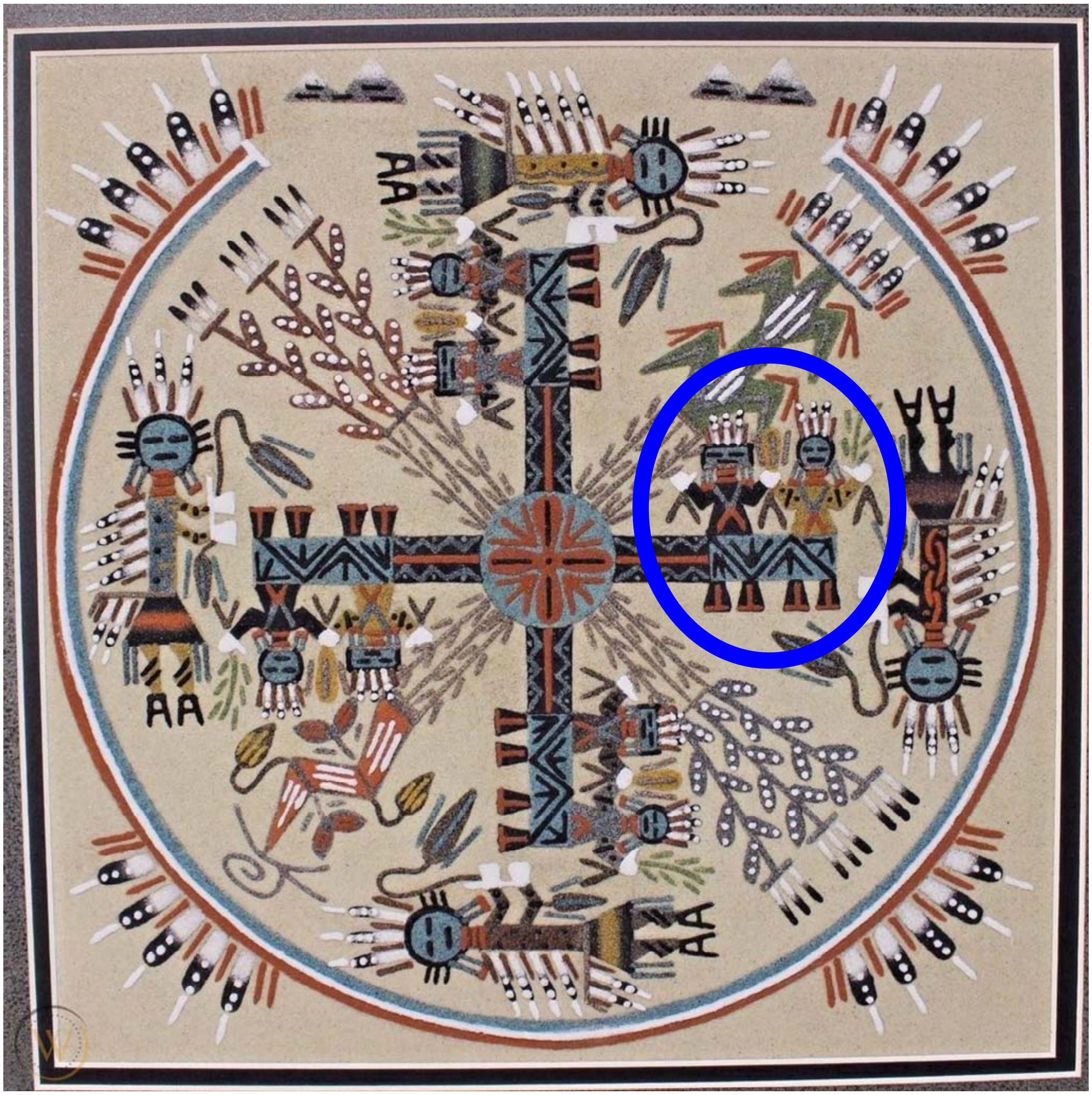
Similarly, rotational symmetry is broken by the lone presence of corn stalk in the upper right corner. Since this image appears only in this one place, any rotation of the sandpainting would be immediately apparent.



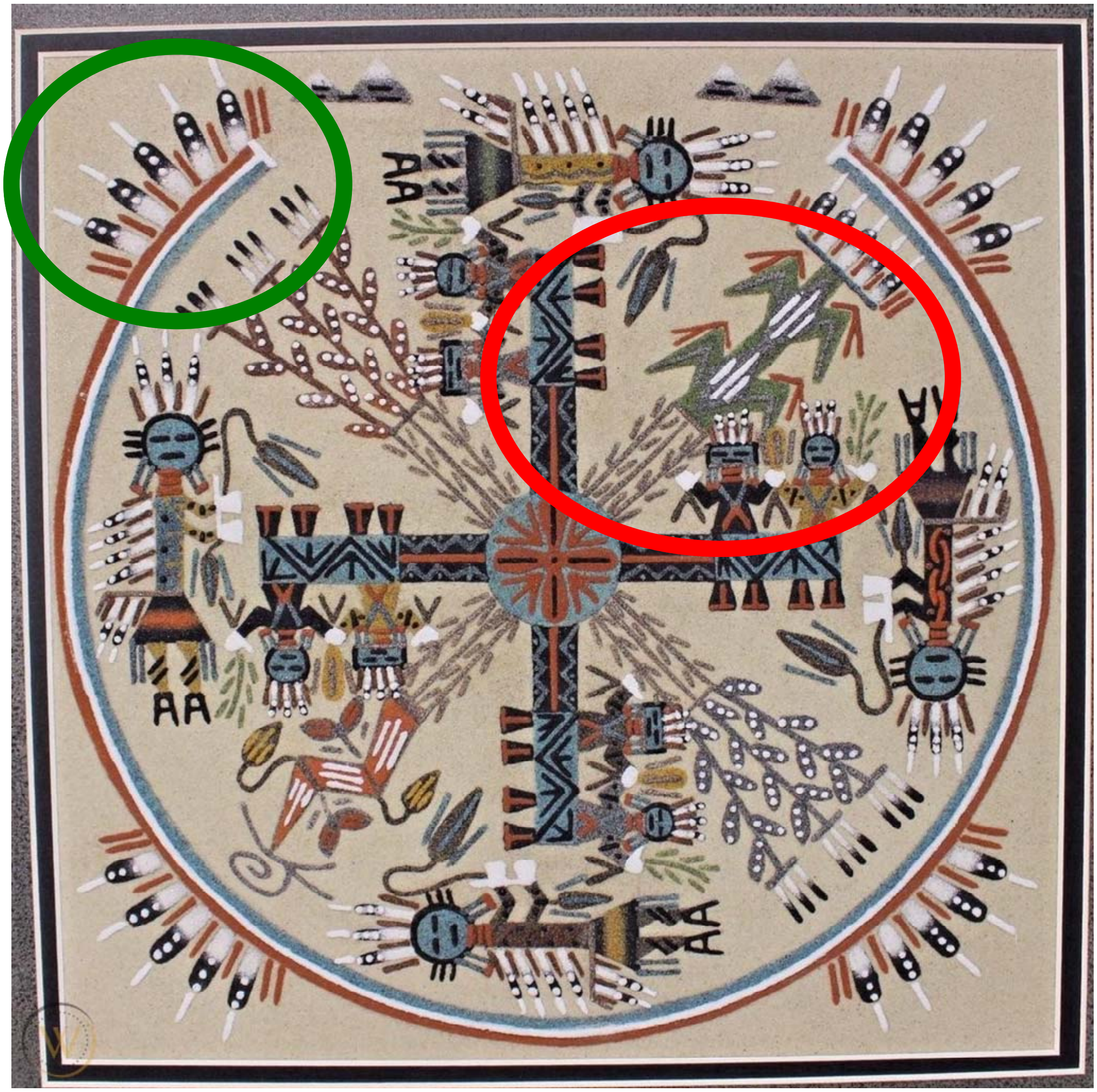
However, in spite of those elements which ruin our perfect symmetry, the image still has what I might call a suggested, intended, or apparent symmetry – namely, D_4, or at the very least, C_4. Additionally, it is interesting how quick our brains are to pick out patterns or symmetries within experiences. I personally suspect that when a brain can identify a pattern that it results in an experience that is easier both to comprehend and remember. It's much easier to conceive of one image and then repeat instead of trying to process a thousand different images at once!



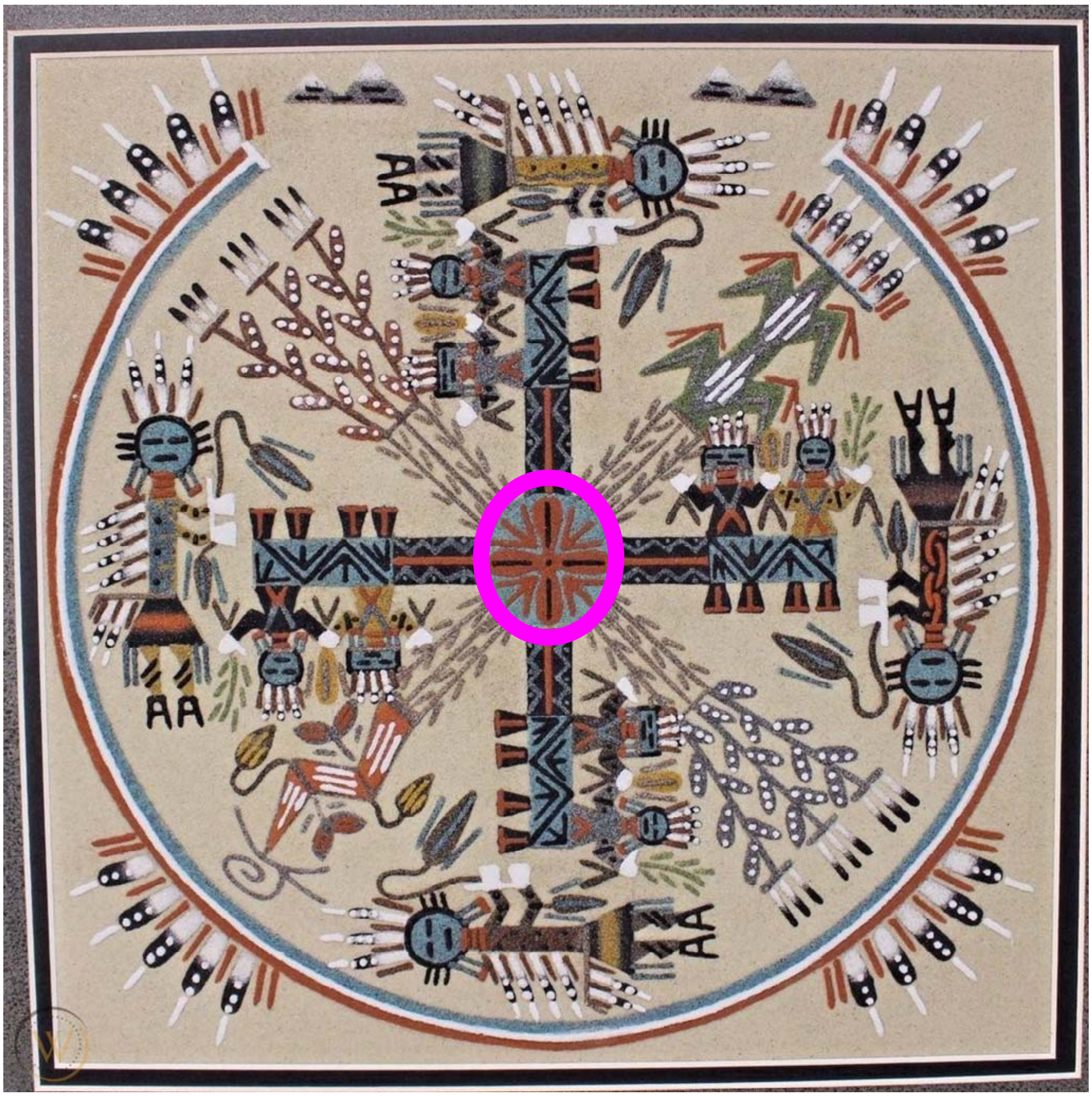
A good example of the brains desire for symmetry is illustrated by the two figures that are circled below. They are obviously not 100% identical to one another, but they do have certain features, such as the arms and feat, that look identical, and these are the features that our brains like to latch onto in order to perceive the apparent symmetry, regardless of how imperfect it may be.



Also, notice in the elements circled below the bilateral symmetry of the corn stalk and the repetition of eagle feathers. It is common in many designs such as this to have symmetries embedded within the symmetries of larger images, and this added complexity enriches our appreciation of such images.



And finally, if we look at the image at the very center of our sandpainting, then we can see that this image does perfectly possess the symmetry that corresponds to D_4, the *dihedral group* of degree 4 that contains 8 elements including 4 rotations and 4 reflections.

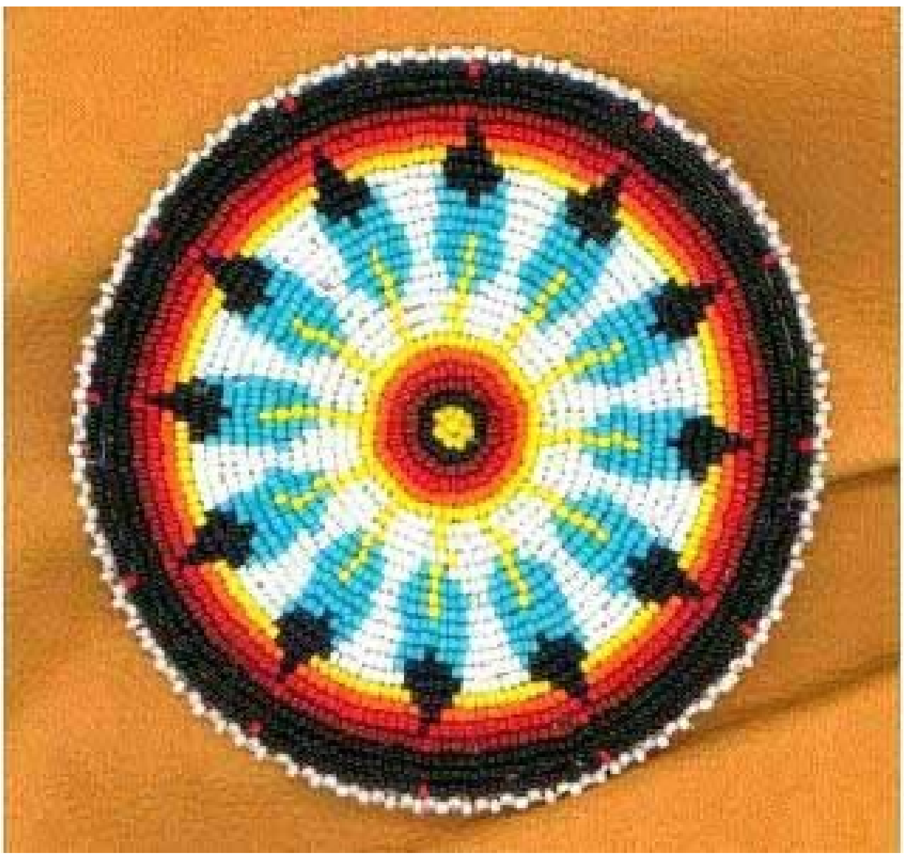


NATIVE AMERICAN MEDALLIONS-1

Below is a fairly common Native American medallion. It has both rotational symmetry and mirror lines. In particular, the rotational symmetry defines a *cyclic group* of order 8, and then there are also 8 axis of reflection. Consequently, the *symmetry group* corresponding to this design is D_8, the *dihedral group* of order 16.

NATIVE AMERICAN MEDALLIONS-2

The medallion below also has both rotational symmetry and mirror lines just like the previous one, but this time the *cyclic group* corresponding to the rotations has order 12 and there are 12 mirror lines. Consequently, the corresponding *symmetry group* is D_{12}, a *dihedral group* that has 24 elements.



NATIVE AMERICAN MEDALLIONS-4

If you look closely at this next medallion, it has rotational symmetry, but no mirror lines. Thus, its *symmetry group* is just C_{12}, the *cyclic group* of order 12.

NATIVE AMERICAN MEDALLIONS-5

Most Native American medallions that I come across have patterns of symmetry that correspond to either a *cyclic group* or a *dihedral group*. That makes this next pattern all the more interesting because it exhibits rotational symmetry of degree 2 as well as a vertical axis of reflection and a horizontal axis of reflection. Consequently, its symmetry corresponds to the *Klein 4-group*.