Lesson 5

DIRECT PRODUCTS

So far we've talked about cyclic groups, dihedral groups, and symmetric groups. Now we're going to learn a standard way to construct new groups from old that's called a direct product of groups. The idea behind it is very simple. We simple take two groups and form coordinate pairs where the first element comes from one group and the second element comes from the other group. Next, we add or multiply group elements by doing it coordinatewise using the addition or multiplication for each individual group.

As an example, let's consider the direct product of the integers modulo 2 with the integers modulo 2. This is essentially the same as the direct product of the cyclic group of order two with itself, but in the integers modulo 2 we tend to specify our operation as addition rather than multiplication. Thus, recall that if $\mathbb{Z}_2 = \{0,1\}$, then addition is defined such that $1+1=0$. Hence, we get the following addition table.

+	**0**	**1**
0	0	1
1	1	0

The general notation for the direct product of $\mathbb{Z}_2$ with $\mathbb{Z}_2$ is $\mathbb{Z}_2 \times \mathbb{Z}_2$, but when both groups are abelian, i.e. comnmutative, you also see it written as $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. We'll stick with the former notation, though. Also, as a set, $\mathbb{Z}_2 \times \mathbb{Z}_2$ is going to consist of all ordered pairs we can form where the first element comes from $\mathbb{Z}_2$ and the second element also comes from $\mathbb{Z}_2$. That's going to give us four ordered pairs in all. In the particular, the elements in this direct product will be $\mathbb{Z}_2 \times \mathbb{Z}_2 = \{(0,0,),(1,0),(0,1),(1,1)\}$. And the multiplication (addition) table for this group is as follows.

+	**(0,0)**	**(1,0)**	**(0,1)**	**(1,1)**
(0,0)	(0,0)	(1,0)	(0,1)	(1,1)
(1,0)	(1,0)	(0,0)	(1,1)	(0,1)
(0,1)	(0,1)	(1,1)	(0,0)	(1,0)
(1,1)	(1,1)	(0,1)	(1,0)	(0,0)

Notice that each non-identity element in this group has order 2. That means that when you multiply (add) any element by itself, you get back the identity which in this case is $(0,0)$. Another way to say that is that each non-identity element generates a cyclic group of order 2. Also, notice that based upon the mirror symmetry with respect to the diagonal, we can definitely say that this group is abelian. However, since this group has four elements and since none of them generate the whole group, this group is not cyclic. In fact, this is the smallest example one can find of an abelian group that is not cyclic. And finally, this group has a special name. It is known as the Klein 4-group. Also, if we replace (0,0) by *e*, (1,0) by *a*, 0,1) by *b*, and (1,1) by *c*, then we can rewrite the multiplication table for our Klein 4-group as follows.

*	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

Now comes the tricky part (or the good part, as I say). Below are the two multiplication tables we've constructed, and on the one hand, since they use different notations, we could say that they are two different groups. But on the other hand, the multiplication tables suggest that the two groups essentially have the same structure. In other words, we can establish a correspondence between the elements in such a way that addition in the first table corresponds to multiplication in the second table. For example, with *a* corresponding to (1,0), *b* corresponding to (0,1), and *c* corresponding to (1,1), we can see from the tables that just as $(1,0)+(0,1)=(1,1)$, so does $a*b=c$. Hence, our two groups are essentially the same, but expressed using different notations. Recall that when this happens, we say that the two groups are <u>isomorphic</u>. That is just a nice word that means "equal shape." Additionally, we can use the multiplication tables below to verify that our correspondence or coding works for other elements, too. Using our coding, we'll always have that a sum of elements in the first group corresponds to a product of elements in the second group.

+	(0,0)	(1,0)	(0,1)	(1,1)
(0,0)	(0,0)	(1,0)	(0,1)	(1,1)
(1,0)	(1,0)	(0,0)	(1,1)	(0,1)
(0,1)	(0,1)	(1,1)	(0,0)	(1,0)
(1,1)	(1,1)	(0,1)	(1,0)	(0,0)

*	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

We can also give a real-world example of a Klein 4-group. Below is a picture of two light switches.

Clearly, we can flip each switch independently of the other. Thus, let's define some operations as follows.

e means that you flip no switches
f_1 means that you flip the first switch
f_2 means that you flip the second switch
$f_1 f_2$ means that you flip both switches

These operations will now generate the following multiplication table for a group. By studying the multiplication table, you can see that this group is isomorphic to the Klein 4-group.

*	e	f_1	f_2	f_1f_2
e	e	f_1	f_2	f_1f_2
f_1	f_1	e	f_1f_2	f_2
f_2	f_2	f_1f_2	e	f_1
f_1f_2	f_1f_2	f_2	f_1	e

Here are another couple of examples of direct products. If we look at $\mathbb{Z}_2 \times \mathbb{Z}_3$, then we are going to get a group with 6 elements since the first group has 2 elements and the second group has 3. Specifically, $\mathbb{Z}_2 \times \mathbb{Z}_3 = \{(0,0,),(1,0),(0,1),(0,2),(1,1),(1,2)\}$. Furthermore, if you do addition modulo 2 with the first coordinate and addition modulo 3 with the second coordinate, then you can verify that (1,1) generates the group. Hence, $\mathbb{Z}_2 \times \mathbb{Z}_3$ is isomorphic to C_6, the cyclic group of order 6, and we usually denote this by writing $\mathbb{Z}_2 \times \mathbb{Z}_3 \cong C_6$. This result also illustrates an important theorem. Namely, if a cyclic group has order *mn* where *m* and *n* are relatively prime (in other words, their only common divisor is 1), then our cyclic group is isomorphic to $\mathbb{Z}_m \times \mathbb{Z}_n$.

We can also form the direct product of more than two groups just by extending our earlier definitions. Thus, for example, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ has 8 elements, and $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 = \{(0,0,0),(1,0,0),(0,1,0),(0,0,1),(1,1,0),(1,0,1),(0,1,1),(1,1,1)\}$. Notice, again, that every non-identity element in this direct product has order 2, and so it is not a cyclic group. However, it is abelian.