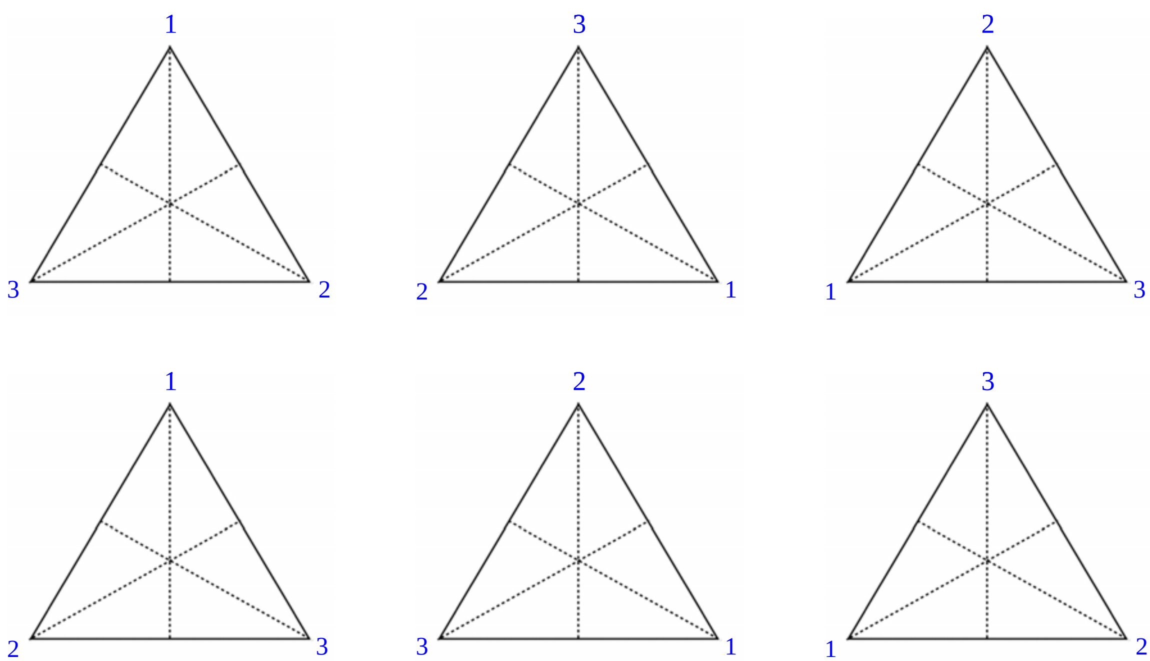
Lesson 3

DIHEDRAL GROUPS

There are a few basic types of groups that we need to be familiar with. The first type, cyclic groups, we just talked about, and we saw that every cyclic group is generated by a single element. And in many ways, cyclic groups or cycles are the building blocks from which all groups are made. The next class of groups we want to learn about are the dihedral groups. They are groups related to the symmetry of a regular polygon, and they are pretty simple.

The word “dihedral” basically means “two faces,” and this term comes from the fact that when we take a regular polygon (i.e. a polygon such that all the sides have the same length), the group corresponding to the symmetry of that regular polygon consists of all the rotations about its center and all reflections about axes of symmetry that “flip” the top and bottom faces while leaving the orientation of the polygon looking unchanged. We’ve already done this for an equilateral triangle, and recall that we found six possible orientations.

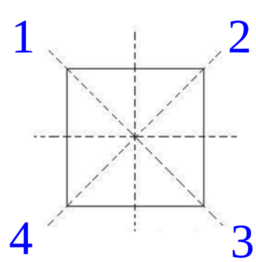


To generate the group corresponding to the symmetries in this triangle, we rotated the triangle through angles that are integer multiples of $120°$, and we did reflections about each of the three axes of symmetry. The result is what we’ll now call the dihedral group D_3 , and notice that the subscript of three refers to the number of sides in our regular polygon, and the order of D_3 is $2 \cdot 3 = 6$. In general, every dihedral group is generated this way. That is, by looking at the permutations of the corner points that can be created either through rotations or reflections about axes of symmetry, and if our regular polygon has n sides, then $|D_n| = 2n$. Furthermore, it can be shown that every dihedral group can be generated by combining the rotations with a flip about a single axis of symmetry. So, for example, let's do this with our equilateral triangle. Let e be the identity (no rotations or flips), let r be a clockwise rotation through an angle of $120°$, and let f always be the flip about the current vertical axis. Then we can write the elements of D_3 as follows, and as usual, we do all multiplication from left to right, though remember that many others do it from right to left. Hence, be prepared to switch when necessary!

$$e = ()$$
$$r = (1,2,3)$$
$$r^2 = (1,3,2)$$
$$f = (2,3)$$
$$fr = (2,3)(1,2,3) = (1,2)$$
$$fr^2 = (2,3)(1,3,2) = (1,3)$$

Notice, too, that $rf = (1,2,3)(2,3) = (1,3) = fr^2$ and $r^2 f = (1,3,2)(2,3) = (1,2) = fr$. In other words, $rf = fr^2$ and $r^2 f = fr$. We might also add to this list that $r^3 = e = f^2$, and equations such as these are often referred to as <u>relations</u> within a group. One way of specifying a group is by giving not only a list of elements that generate the group through the various products that can be formed, but also by including a list of equations or relations that, along with the generators, define that group.

Below is a square with the corner vertices labeled, and we can now easily list all the elements in the dihedral group D_4 in terms of combinations of clockwise rotations through angles of $90°$ and flips about the vertical axis. And we can represent each group element as a permutation of the vertices. Notice that the group generated by these movements has order $|D_4| = 2 \cdot 4 = 8$.



$e = ()$
$r = (1,2,3,4)$
$r^2 = (1,3)(2,4)$
$r^3 = (1,4,3,2)$
$f = (1,2)(3,4)$
$fr = (1,2)(3,4)(1,2,3,4) = (1,3)$
$fr^2 = (1,2)(3,4)(1,3)(2,4) = (1,4)(2,3)$
$fr^3 = (1,2)(3,4)(1,4,3,2) = (2,4)$

Notice also that,

$rf = (1,2,3,4)(1,2)(3,4) = (2,4) = fr^3$
$r^2 f = (1,3)(2,4)(1,2)(3,4) = (1,4)(3,2) = fr^2$
$r^3 f = (1,4,3,2)(1,2)(3,4) = (1,3) = fr$

Hence, we have the following relations in this group.

$r^4 = e$
$f^2 = e$
$rf = fr^3$
$r^2 f = fr^2$
$r^3 f = fr$

Additionally, below is a multiplication table for D_4. You might need a magnifying glass!

*	()	(2,4)	(1,2)(3,4)	(1,2,3,4)	(1,3)	(1,3)(2,4)	(1,4,3,2)	(1,4)(2,3)
()	()	(2,4)	(1,2)(3,4)	(1,2,3,4)	(1,3)	(1,3)(2,4)	(1,4,3,2)	(1,4)(2,3)
(2,4)	(2,4)	()	(1,2,3,4)	(1,2)(3,4)	(1,3)(2,4)	(1,3)	(1,4)(2,3)	(1,4,3,2)
(1,2)(3,4)	(1,2)(3,4)	(1,4,3,2)	()	(1,3)	(1,2,3,4)	(1,4)(2,3)	(2,4)	(1,3)(2,4)
(1,2,3,4)	(1,2,3,4)	(1,4)(2,3)	(2,4)	(1,3)(2,4)	(1,2)(3,4)	(1,4,3,2)	()	(1,3)
(1,3)	(1,3)	(1,3)(2,4)	(1,4,3,2)	(1,4)(2,3)	()	(2,4)	(1,2)(3,4)	(1,2,3,4)
(1,3)(2,4)	(1,3)(2,4)	(1,3)	(1,4)(2,3)	(1,4,3,2)	(2,4)	()	(1,2,3,4)	(1,2)(3,4)
(1,4,3,2)	(1,4,3,2)	(1,2)(3,4)	(1,3)	()	(1,4)(2,3)	(1,2,3,4)	(1,3)(2,4)	(2,4)
(1,4)(2,3)	(1,4)(2,3)	(1,2,3,4)	(1,3)(2,4)	(2,4)	(1,4,3,2)	(1,2)(3,4)	(1,3)	()

Recall now that since $|D_4| = 2 \cdot 4 = 8$, every subgroup of D_4 must have an order that is a divisor of 8, and hence, the only possible orders for subgroups are 1, 2, 4, or 8. As it turns out, D_4 has 10 subgroups, and since D_4 has 8 elements, that means that we can generate eight cyclic groups from those elements. However, as we'll see, some of those cyclic subgroups are the same. And another subgroup is D_4, the entire group which, in this case, is not cyclic. We can list the cyclic subgroups generated by the eight elements as follows.

$\langle e \rangle = \langle () \rangle = \{()\}$
$\langle r \rangle = \langle (1,2,3,4 \rangle = \{(),(1,2,3,4),(1,3)(2,4),(1,4,3,2\}$
$\langle r^2 \rangle = \langle (1,3)(2,4) \rangle = \{(),(1,3)(2,4)\}$
$\langle r^3 \rangle = \langle (1,4,3,2) \rangle = \{(),(1,2,3,4),(1,3)(2,4),(1,4,3,2\}$
$\langle f \rangle = \langle (1,2)(3,4) \rangle = \{(),(1,2)(3,4)\}$
$\langle fr \rangle = \langle (1,3 \rangle = \{(),(1,3)\}$
$\langle fr^2 \rangle = \langle (1,4)(2,3) \rangle = \{(),(1,4)(2,3)\}$
$\langle fr^3 \rangle = \langle (2,4) \rangle = \{(),(2,4)\}$

Additionally,

$$D_4 = \langle f, r \rangle = \langle (2,4) \rangle = \{(),(1,2,3,4),(1,3)(2,4),(1,4,3,2),(1,2)(3,4),(1,3),(1,4)(2,3),(2,4)\}$$

Since $\langle r \rangle = \{(),(1,2,3,4),(1,3)(2,4),(1,4,3,2\} = \langle r^3 \rangle$, this list of cyclic groups plus D_4 gives us only eight distinct subgroups, and so we still need to find two more. The two additional groups are,

$\langle f, fr^2 \rangle = \langle (1,2)(3,4),(1,4)(2,3) \rangle = \{(),(1,2)(3,4),(1,3)(2,4),(1,4)(2,3)\}$
$\langle r^2, fr \rangle = \langle (1,3)(2,4),(1,4)(2,3) \rangle = \{(),(2,4),(1,3),(1,3)(2,4)\}$

Again, notice that all of these subgroups have orders which divide 8, the order of D_4.