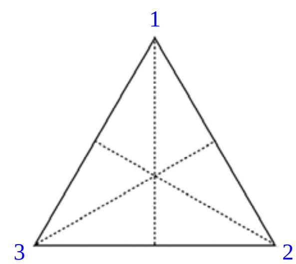
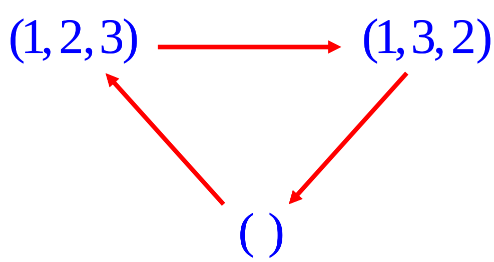
# Lesson 7

## CYCLE GRAPHS

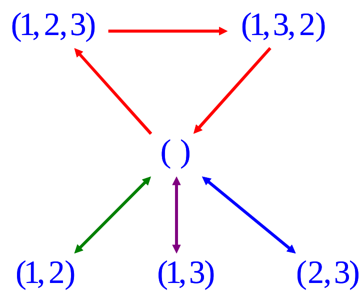
There are a variety of ways to graphically depict characteristics of a group, and one of those is through what we call cycle graphs. A cycle graph is just a diagram that illustrates the cyclic subgroup generated by each element, and if two groups have different cycle graphs, then they obviously can't be isomorphic, i.e. essentially the same group with different labels for the elements. On the other hand, it is possible for two groups (unfortunately) to have identical cycle graphs and, yet, not be isomorphic. However, this doesn't happen until we get to groups of order 16, and so the groups of smaller order that we'll examine will all have different cycle graphs. Let's illustrate by examining some of the groups that can be associated with our equilateral triangle.

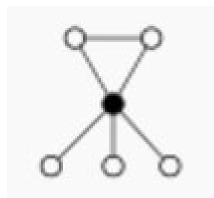


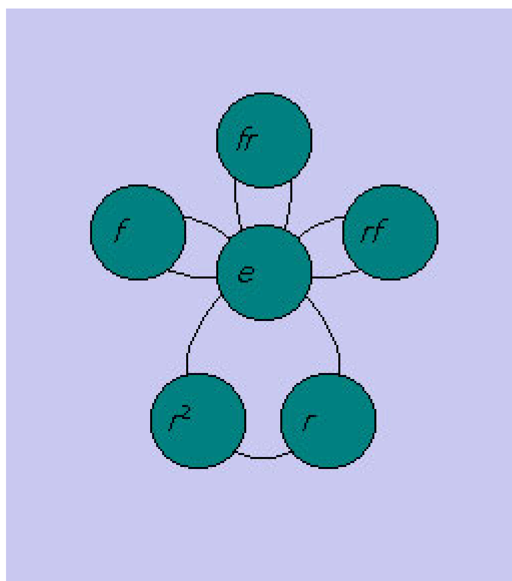

One of the simpler groups we could associate with this diagram is the cyclic group of order 3, $C_3 = \{(\,),(1,2,3),(1,3,2)\}$, and below is how I denote its cycle graph.



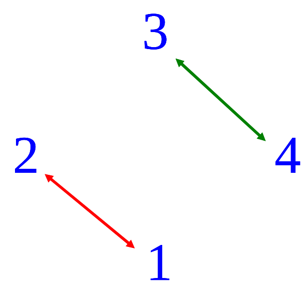

This graph is pretty simple, and notice that since $(1,3,2)$ is already part of the cycle generated by $(1,2,3)$, we don't create a second cycle drawing just for it. That helps to keep things simple. Also, as our groups become more intricate, so do the corresponding cycle graphs. Below now are three different versions of a cycle graph for $D_3 \cong S_3 = \{(\,),(1,2,3),(1,3,2),(1,2),(1,3),(2,3)\}$ Notice that every element of the group must appear in one of the cycles.



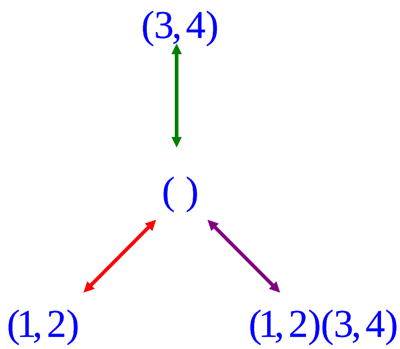





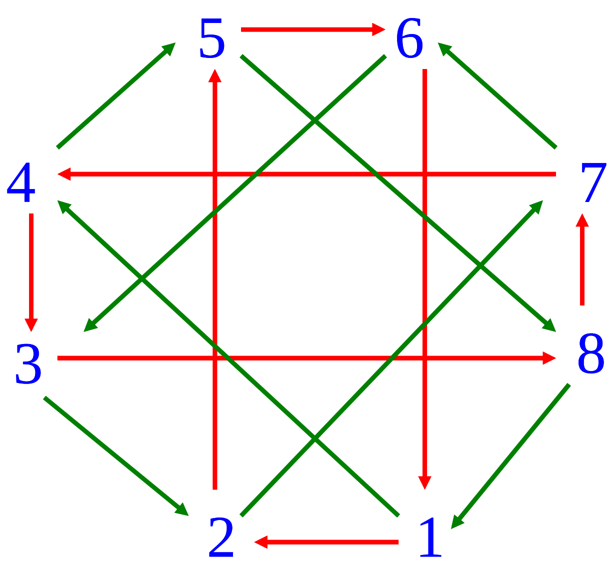

We'll do our cycle graphs following the first diagram where each element is given as a permutation. And now, let's look at the cycle graph for the Klein 4-group. Recall that this group can be represented by $\mathbb{Z}_2 \times \mathbb{Z}_2 = \{(0,0),(1,0),(0,1),(1,1)\}$ where the elements represent coordinates instead of permutations, and the operation is addition modulo 2. However, we can also represent the Klein 4-group as the group of permutations $C_2 \times C_2 = \{(\ ),(1,2),(3,4),(1,2)(3,4)\}$ that acts on the generator diagram below.




For this group, our cycle graph is as follows. Notice that each element generates a subgroup of order 2.




And now, I'd like to introduce you to a very special group of eight elements called the Quaternion Group, *Q*. Here is a generator diagram for the group followed by the group itself. The generators for this group are $(1,2,5,6)(3,8,7,4)$ and $(1,4,5,8)(2,7,6,3)$.



And here is a list of the elements in the group *Q* along with the corresponding cycle graph. From this cycle graph we can see that six of the elements generate groups of order 4, one element generates a group of order 2, and the identity generates a group of order 1. The element of order 2 is the permutation (1,5)(2,6)(3,7)(4,8), and we can see that it is also contained in each cycle of order 4. Very interesting!

Q = { (), (1,2,5,6)(3,8,7,4), (1,3,5,7)(2,4,6,8), (1,4,5,8)(2,7,6,3), (1,5)(2,6)(3,7)(4,8), (1,6,5,2)(3,4,7,8), (1,7,5,3)(2,8,6,4), (1,8,5,4)(2,3,6,7) }