Homogeneous Second Order Linear Differential Equations With Constant Coefficents

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$$

How do we solve this equation?

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

Let's just make a lucky guess and assume that the solution has the form e^{rx}, and let's see what the consequences are.

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

Consequences:

$$y = e^{rx} \Rightarrow$$

$$a\frac{d^2 y}{dx^2} + b\frac{dy}{dx} + cy = ar^2 e^{rx} + bre^{rx} + ce^{x}$$

$$= \left(ar^2 + br + c\right)e^{rx} = 0$$

$$\Rightarrow ar^2 + br + c = 0$$

The expression ar^2+br+c is called the characteric polynomial, and to find the solutions to the differential equation, we must solve the equation $ar^2+br+c=0$.

$$ar^2+br+c=0$$

$$r=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Example 1: $ar^2+br+c=0$ has two distinct real roots

$$\frac{d^2y}{dx^2}-5\frac{dy}{dx}+6y=0$$

$$\Rightarrow r^2-5r+6=0$$

$$\Rightarrow (r-2)(r-3)=0$$

$$\Rightarrow r=2 \text{ or } r=3$$

$$\Rightarrow y=C_1e^{2x}+C_2e^{3x}$$

Example 2: $ar^2+br+c=0$ has one repeated real root

$$\frac{d^2y}{dx^2}-6\frac{dy}{dx}+9y=0$$

$$\Rightarrow r^2-6r+9=0$$

$$\Rightarrow (r-3)(r-3)=(r-3)^2=0$$

$$\Rightarrow \quad r=3$$

$$\Rightarrow y=e^{3x} \quad \text{is a solution}$$

Example 2: At this point we have one solution, but we need a second, linearly independent solution. Again, we'll make a lucky guess that $y=xe^{3x}$ is also a solution.

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 0, \quad y = e^{3x} \quad \text{is a solution}$$

Verification:

$$y = xe^{3x} \Rightarrow$$

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y$$

$$= (9xe^{3x} + 6e^{3x}) - 6(3xe^{3x} + e^{3x}) + 9xe^{3x}$$

$$= 9xe^{3x} + 6e^{3x} - 18xe^{3x} - 6e^{3x} + 9xe^{3x}$$

$$= 18xe^{3x} - 18xe^{3x} + 6e^{3x} - 6e^{3x} = 0$$

Thus, the general solution is:

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$$

$$y = C_1e^{3x} + C_2xe^{3x}$$

Example 3: ar^2+br+c=0 has two distinct complex roots

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 13y = 0$$

$$\Rightarrow r^2 - 4r + 13 = 0$$

$$\Rightarrow r = \frac{4 \pm \sqrt{16-52}}{2} = \frac{4 \pm 6i}{2} = 2 \pm 3i$$

$$\Rightarrow r = 2 + 3i \quad \text{or} \quad r = 2 - 3i$$

$$\Rightarrow y = C_1 e^{2+3i} + C_2 e^{2-3i}$$

The only problem with solutions such as these is that they are complex, and we want solutions that are real. Let's see if we can do some manipulation using Euler's Formula.

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 13y = 0$$

$$y = C_1 e^{(2+3i)x} + C_2 e^{(2-3i)x}$$

Euler's Formula: (Part 1)

$$e^{(2+3i)x} = e^{2x+3xi} = e^{2x}e^{3xi}$$

$$= e^{2x}(\cos 3x + i\sin 3x) = e^{2x}\cos 3x + ie^{2x}\sin 3x$$

$$e^{(2-3i)x} = e^{2x-3xi} = e^{2x}e^{-3xi}$$

$$= e^{2x}(\cos(-3x) + i\sin(-3x)) = e^{2x}\cos 3x - ie^{2x}\sin 3x$$

Euler's Formula: (Part 2) Now do a little addition to get another solution.

$$e^{(2+3i)x} + e^{(2-3i)x}$$

$$= \left[e^{2x} \cos 3x + ie^{2x} \sin 3x \right] + \left[e^{2x} \cos 3x - ie^{2x} \sin 3x \right]$$

$$= 2e^{2x} \cos 3x$$

Euler's Formula: (Part 2) Notice that if we divide this solution by 2, then we still have a solution.

$$\frac{e^{(2+3i)x}+e^{(2-3i)x}}{2}$$

$$=\frac{\left[e^{2x}\cos 3x+ie^{2x}\sin 3x\right]+\left[e^{2x}\cos 3x-ie^{2x}\sin 3x\right]}{2}$$

$$=\frac{2e^{2x}\cos 3x}{2}$$

$$=e^{2x}\cos 3x$$

<u>Euler's Formula:</u> (Part 3) Now do a little subtraction to get a second, independent solution.

$$e^{(2+3i)x} - e^{(2-3i)x}$$

$$= \left[e^{2x}\cos 3x + ie^{2x}\sin 3x \right] - \left[e^{2x}\cos 3x - ie^{2x}\sin 3x \right]$$

$$= 2ie^{2x}\sin 3x$$

<u>Euler's Formula:</u> (Part 3) Notice that if we divide this solution by 2i, then we still have a solution.

$$\frac{e^{(2+3i)x} - e^{(2-3i)x}}{2i}$$

$$= \frac{\left[e^{2x}\cos 3x + ie^{2x}\sin 3x\right] - \left[e^{2x}\cos 3x - ie^{2x}\sin 3x\right]}{2i}$$

$$= \frac{2ie^{2x}\sin 3x}{2i}$$

$$= e^{2x}\sin 3x$$

Thus, the general solution can be written as follows:

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 13y = 0$$

$$y = C_1e^{2x}\cos 3x + C_2e^{2x}\sin 3x$$

$$= e^{2x}(C_1\cos 3x + C_2\sin 3x)$$

Summary:

Equation: $a\dfrac{d^2y}{dx^2} + b\dfrac{dy}{dx} + c = 0$

Characteristic Polynomial: $ar^2 + br + c = 0$ with roots r_1 and r_2

Case 1: r_1 and r_2 distinct real numbers

$$y = C_1 e^{r_1 x} + C_2 e^{r_2 x}$$

Case 2: $r_1 = r_2$ and real

$$y = C_1 e^{r_1 x} + C_2 x e^{r_1 x}$$

Case 3: $r_1 = u + vi$ and $r_1 = u - vi$, complex

$$y = C_1 e^{ux} \cos vx + C_2 e^{ux} \sin vx = e^{ux}(C_1 \cos vx + C_2 \sin vx)$$

By the way, there's another way to look at the solution for Case 3.

Case 3: $r_1 = u + vi$ and $r_1 = u - vi,$ complex

$$y = C_1 e^{ux} \cos vx + C_2 e^{ux} \sin vx = e^{ux}(C_1 \cos vx + C_2 \sin vx)$$

The portion in parentheses can be thought of as a dot product.

Case 3: $r_1 = u + vi$ and $r_1 = u - vi,$ complex

$$y = C_1 e^{ux} \cos vx + C_2 e^{ux} \sin vx = e^{ux} (C_1 \cos vx + C_2 \sin vx)$$

$$C_1 \cos vx + C_2 \sin vx = (C_1 \,\hat{i} + C_2 \,\hat{j}) \bullet (\cos vx \,\hat{i} + \sin vx \,\hat{j})$$

If *phi* is the angle that the first vector makes with the positive *x*-axis and *vx* is the angle that the second vector makes, then we can rewrite this dot product as follows:

Case 3: $r_1 = u + vi$ and $r_1 = u - vi$, complex

$$y = C_1 e^{ux} \cos vx + C_2 e^{ux} \sin vx = e^{ux}(C_1 \cos vx + C_2 \sin vx)$$

$$C_1 \cos vx + C_2 \sin vx = (C_1 \hat{i} + C_2 \hat{j}) \bullet (\cos vx \hat{i} + \sin vx \hat{j})$$

$$= \left\| C_1 \hat{i} + C_2 \hat{j} \right\| \cdot \left\| \cos vx \hat{i} + \sin vx \hat{j} \right\| \cdot \cos(vx - \varphi)$$

$$= \sqrt{C_1^{\,2} + C_2^{\,2}} \cdot \cos(vx - \varphi)$$

For specific values of C_1 and C_2, we can always find *phi* by using the inverse tangent function along with some standard trigonometry.

Case 3: $r_1 = u + vi$ and $r_1 = u - vi$, complex

$$y = C_1 e^{ux}\cos vx + C_2 e^{ux}\sin vx = e^{ux}(C_1\cos vx + C_2\sin vx)$$

$$\begin{aligned}
C_1\cos vx + C_2\sin vx &= (C_1\,\hat{i} + C_2\,\hat{j})\bullet(\cos vx\,\hat{i} + \sin vx\,\hat{j}) \\
&= \left\| C_1\,\hat{i} + C_2\,\hat{j} \right\| \cdot \left\| \cos vx\,\hat{i} + \sin vx\,\hat{j} \right\| \cdot \cos(vx - \varphi) \\
&= \sqrt{C_1{}^2 + C_2{}^2} \cdot \cos(vx - \varphi)
\end{aligned}$$

The bottom line is that the solution in the third case can be written in terms of a simple variant of the cosine function, and this tells us something about the amplitude, period, and phase shift of the solution.

Equation: $a\dfrac{d^2y}{dx^2} + b\dfrac{dy}{dx} + c = 0$

Characteristic Polynomial: $ar^2 + br + c = 0$ with roots r_1 and r_2

Case 3: $r_1 = u + vi$ and $r_1 = u - vi$, complex

$$y = C_1 e^{ux} \cos vx + C_2 e^{ux} \sin vx = e^{ux}(C_1 \cos vx + C_2 \sin vx)$$

$$e^{ux}\sqrt{C_1{}^2 + C_2{}^2} \cdot \cos(vx - \varphi)$$