A THIRD APPLICATION

Discussion: Below, we define an isomorphism from a group onto itself as an automorphism, and we show that the operation of conjugation, that we introduced back in our section on beginning group theory, results in a very important automorphism that we call an inner automorphism.

Definition: An isomorphism from a group *G* onto itself is called an automorphism.

Theorem: Let *G* be a group, let $g \in G$, and defined a function $f_g : G \to G$ by $f_g(a) = g^{-1}ag$ for every $a \in G$. Then f_g is an automorphism.

Proof: Let *G* be a group, let $g \in G$, and define a function $f_g : G \to G$ by $f_g(a) = g^{-1}ag$ for every $a \in G$. To show that f_g is an automorphism, we need to show that it is a homomorphism, it's onto, and it's one-to-one. To show that it's a homomorphism, let $a, b \in G$. Then $f_g(a) f_g(b) = g^{-1}ag \cdot g^{-1}bg = g^{-1}a \cdot e \cdot bg = g^{-1}(ab)g = f_g(ab)$. To show that it's onto, let $a \in G$. Then gag^{-1} is also an element of *G*, and $f_g(gag^{-1}) = g^{-1}(gag^{-1})g = e \cdot a \cdot e = a$. And finally, to show that $f_g : G \to G$ is one-to-one, suppose $a \in Ker(f_g)$. Then $f_g(a) = e \Rightarrow g^{-1}ag = e \Rightarrow a = geg^{-1} = e$. Hence, $Ker(f_g)$ consists only of *e*, and the homomorphism is one-to-one. Therefore, $f_g : G \to G$ defined by $f_g(a) = g^{-1}ag$ is an automorphism. Furthermore, this particular type of automorphism is called an inner automorphism.

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