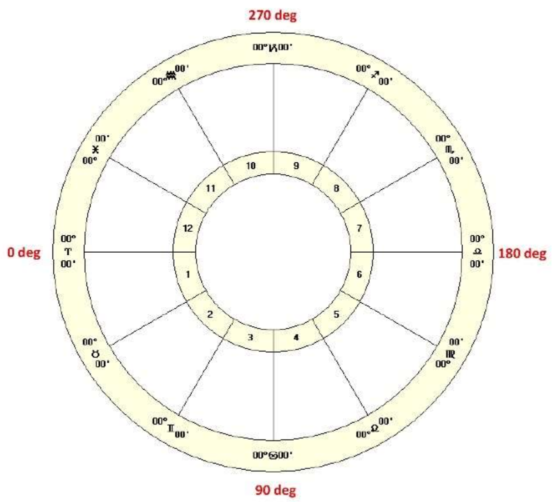
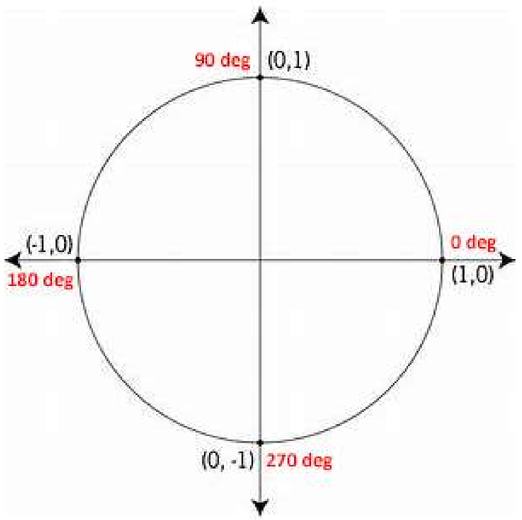
ASTROLOGY AND COMPLEX NUMBERS

A rather surprising development is the realization that complex numbers can be used to model certain aspects of harmonic charts in astrology. For example, recall that the 4th harmonic angles in a horoscope are those that are integer multiples of $90°\, modulo\, 360°$ since $\frac{360°}{4} = 90°$. Hence, the distinct 4th harmonic angles are $0°, 90°, 180°$, and $270°$, and these angles are equally spaced around the zodiacal circle. A similar situation occurs in the realm of complex numbers where the number 1 always has *n* distinct complex roots that all lie on a circle of radius 1 with center at the origin (called the *unit circle*) and like the 4th harmonic angles they are equally spaced at $0°, 90°, 180°$, and $270°$. More generally, if *n* is a natural number, then the n^{th} harmonic angles θ such that $0° \leq \theta < 360°$ are $k\left(\frac{360°}{n}\right)$ where $k = 0,1,2, \ldots, n-1$, and these are the exact same angles that are used in complex numbers to represent the n^{th} *roots of unity*! For example, below are diagrams of the zodiac and the unit circle showing the placements of the 4th harmonic angles in each. Except for the arbitrary placement of 0° as a starting point, the diagrams are the same.



The Zodiac



Unit Circle in the Complex Plane

A complex number has two basic forms. First, there is its *rectangular form* which looks like $z = a + bi$, and second, there is its *polar form* which appears as $z = re^{i\theta}$ where *i* is the square root of -1, *r* is the number's distance from the origin in the complex plane, and θ is the angle that is made with the positive real number axis. Additionally, these two forms are related by the formula $re^{i\theta} = r(cos\theta + isin\theta)$. This latter expression in parentheses is often abbreviated by mathematicians as $cis(\theta)$. Furthermore, the polar form will make it easy to prove various properties of complex numbers that also apply to harmonic charts in astrology. We begin with the following.

Definition: If n is a natural number, then the angle $\frac{360°}{n}$ will be referred to as the n^{th} *harmonic* or the *fundamental* or *root* n^{th} *harmonic angle*. Furthermore, for any *fundamental* or *root* n^{th} *harmonic angle*, by the *family of all distinct* n^{th} *harmonic angles* we will mean the set $\left\{k\left(\frac{360°}{n}\right) modulo\ 360° \middle|\ k \in \mathbb{Z}\right\}$, and the number n that we divide 360° by will also frequently be referred to as the n^{th} *harmonic* just as is the corresponding angle $\frac{360°}{n}$. Also, note that if $n = 4$ and $k = 2$, then the $\frac{2}{4}$ harmonic angle is $\frac{2}{4}(360°) = 180°$, and this is exactly the same as the $\frac{1}{2}$ harmonic angle since $\frac{1}{2}(360°) = 180°$. Thus, even though $180°$ is both a $\frac{1}{2}$ and a $\frac{2}{4}$ harmonic angle, most of the time we will reduce a fraction to its lowest terms when talking about what kind of harmonic angle it is. Additionally, even though an angle like $180°$ is both a 2nd harmonic angle and a 4th harmonic angle (among others), we will usually identify this and other angles with the smallest natural number harmonic that applies.

Example: If, for instance, we want to examine or discuss the 4^{th} harmonic, then the fundamental or root 4^{th} harmonic angle is $\frac{360°}{4} = 90°$ and the family or set of distinct 4^{th} harmonic angles is $\{0°, 90°, 180°, 270°\}$. However, of these angles, $\frac{0}{4}, \frac{1}{4}, \frac{2}{4}$, and $\frac{3}{4}$ $times$ $360°$, we usually reduce ones like $\frac{2}{4}$ to $\frac{1}{2}$ and refer to it primarily as a $\frac{1}{2}$ harmonic angle or a 2nd harmonic.

Comment: There are two ways to find the position of an n^{th} harmonic. We can find, for example, the value of θ $modulo$ $\frac{360°}{n}$ and then multiply this result by n, or we can multiply θ by n and then express the result $modulo$ $360°$. Recall that one way to find a value of an angle θ $modulo$ α is to subtract a multiple of α from θ that yields a result that is less than α but greater than or equal to zero $(0 \leq \theta - k \cdot \alpha < \alpha)$. However, we can easily see from the math that $n\left(\theta\ modulo\ \frac{360°}{n}\right) = n\left(\theta - k \cdot \frac{360°}{n}\right) = n\theta - k \cdot 360° = n\theta\ modulo\ 360°$, and this means that the we can compute our final value using either the first formula, $n\left(\theta\ modulo\ \frac{360°}{n}\right)$, or the last formula, $n\theta\ modulo\ 360°$. In practice, most people usually find the second formula easier since all one has to do is to first multiply θ by the appropriate harmonic and then express the result $modulo$ $360°$.

Summary: Before we proceed to the proofs, below is a listing of some of our main results. Many of these are probably known either consciously or through some other level of experience by those astrologers who specialize in harmonics, but the point is that we are providing more rigorous proofs of these things and by fleshing out the underlying mathematical theory, other things that are currently not so evident will gradually become clear. So with that said, here are some of our results.

1. If $j = mn$, then the j^{th} harmonic is equal to the m^{th} harmonic of the n^{th} harmonic.
2. If θ is an n^{th} harmonic angle equal to $k\left(\frac{360°°}{n}\right)$ for a natural number n where $k = 0,1,2,\ldots, n-1$, then any integer multiple of θ $modulo$ $360°$ is also an n^{th} harmonic angle.
3. If θ_1 and θ_2 are angles such that $\theta_2 - \theta_1$ is an n^{th} harmonic angle and if k is a natural number, then the difference between the corresponding angles in the k^{th} harmonic is also an n^{th} harmonic angle.
4. If θ is an n^{th} harmonic angle for some natural number n, then $n\theta = 0°\ modulo\ 360°$. In other words, θ is equivalent to 0° in an n^{th} harmonic chart.
5. If θ is equivalent to 0° in an n^{th} harmonic chart, then θ is an n^{th} harmonic angle.
6. The angle 0° is equivalent to 0° in all n^{th} harmonic charts.

7. If n and m are natural numbers and if n is a factor of m, then each of the n^{th} harmonic angles is equivalent to $0°\ modulo\ 360°$ in the m^{th} harmonic chart.
8. If m is equal to a natural number power of $n+1$, then each of the n^{th} harmonic angles is an unchanged fixed point when the original chart is transformed into the m^{th} harmonic.
9. If θ is a fixed point in an $(n+1)^{th}$ harmonic chart, then θ is an n^{th} harmonic angle.
10. If n and m are natural numbers and if n is a factor of m, then each of the n^{th} harmonic angles is equivalent to $0°\ modulo\ 360°$ in the m^{th} harmonic chart.
11. If m is not a power of $n+1$ and if n is not a factor of m, then the m^{th} harmonic will produce a permutation of the n^{th} harmonic angles.
12. If θ is a root m^{th} harmonic angle for some natural number m and a natural number n divides m, then the root n^{th} harmonic angle is a natural number multiple of the root m^{th} harmonic angle.
13. If an angle θ can be written as $\theta = \theta_1 + \theta_2$ where θ_1, θ_2 are n^{th} and m^{th} harmonic angles for natural numbers n and m, respectively, and if k is an integer, then $k\theta$ can also be written as the sum of n^{th} and m^{th} harmonic angles.
14. If $z = re^{i\theta}$ and $\theta = \theta_1 + \theta_2$ where θ_1, θ_2 are n^{th} and m^{th} harmonic angles for natural numbers n and m, and if k is an integer, then z^k can be written as a product of complex numbers with angles that are n^{th} and m^{th} harmonic angles.
15. If $\theta = \theta_1 + \theta_2$ where θ_1, θ_2 are n^{th} and m^{th} harmonic angles for natural numbers n and m and if $q = lcm(n, m)$ is the least common multiple of m and n, then θ is a q harmonic angle.
16. If an angle is the sum of n^{th} and m^{th} harmonic angles for some natural numbers n and m with θ_1 and θ_2 being the respective n^{th} and m^{th} root harmonic angles and if $q = lcm(n, m)$ is the least common multiple of m and n, then $\theta = \frac{360°}{q}$, the root q^{th} harmonic angle, is the greatest common divisor of θ_1 and θ_2. In symbols, $\theta = gcd(\theta_1, \theta_2)$.
17. If an angle is the sum of two angles θ_1 and θ_2 where θ_1, θ_2 are n^{th} and m^{th} harmonic angles, respectively, for natural numbers n and m and if $q = lcm(n, m)$ and $\theta = gcd(\theta_1, \theta_2)$, then $\frac{360°}{q} = \theta = \theta_1 + \theta_2$ and $n\theta$ is an m^{th} harmonic angle.
18. If m and n are natural numbers with no common divisor other than 1 and if $q = lcm(m, n) = mn$ and $\theta = \frac{360°}{q}$, then $n\theta$ is an m^{th} harmonic angle and $m\theta$ is an n^{th} harmonic angle.
19. If m and n are natural numbers with no common divisor other than 1 and if $q = lcm(m, n) = mn$ and j, k are natural numbers such that $j = kq = kmn$ and if $\theta = \frac{360°}{j}$, then $kn\theta$ is an m^{th} harmonic angle and $km\theta$ is an n^{th} harmonic angle.
20. If an angle θ as measured in degrees is a rational fraction of 360° of the form $\frac{360°}{f/g} = \frac{g(360°)}{f}$ where $f, g \in \mathbb{N}$, then $f\theta = 0°\ modulo\ 360°$ and $(f+1)\theta = \theta\ modulo\ 360°$.
21. If $\theta = \frac{360°}{f}$ where f is irrational, then θ is irrational, and there is no natural number g such that $g\theta = 0°\ modulo\ 360°$.
22. If $\theta = \frac{360°}{f}$ where f is irrational, then there is no natural number $g > 1$ such that $g\theta = \theta\ modulo\ 360°$.
23. Let $\theta = k\left(\frac{360°}{n}\right)$ be a nonzero n^{th} harmonic angle, and let m be a natural number. If $\theta = k\left(\frac{360°}{n}\right)$ is fixed in the m^{th} harmonic, then all n^{th} harmonic angles are fixed in the m^{th} harmonic.

24. The n^{th} roots of unity form a finite cyclic group of order n under addition that is isomorphic to $\mathbb{Z}_n$.

And now, let's do some proofs!

Theorem: If $j = mn$, then the j^{th} harmonic is equal to the m^{th} harmonic of the n^{th} harmonic.

Proof: If θ is an angle, then $j\theta \; modulo \; 360°$ is the j^{th} harmonic of that angle. But since $j\theta = m(n\theta) \; modulo \; 360°$, it follows immediately that the j^{th} harmonic is equal to the m^{th} harmonic of the n^{th} harmonic.

□

Comment: This theorem greatly simplifies our understanding of things by assuring us that if we start, for example, with a 3^{rd} harmonic chart and then take the 2^{nd} harmonic of that chart, then the end result is the same as taking the 6^{th} harmonic of the original chart. Thus, if you are looking at a 3^{rd} harmonic chart and notice an opposition that you want to convert to a conjunction by computing the 2^{nd} harmonic of the 3^{rd} harmonic, then this gives the same result as taking the 6^{th} harmonic of your original chart!

Theorem: If z_1, z_2 are complex numbers, then the magnitude of their product is equal to the product of their magnitudes, and the angle of their product is equal to the sum of their angles $modulo \; 360°$.

Proof: This is a well-known result that we are presenting for the sake of completeness. Thus, if our complex numbers have the polar forms $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$ where r_1 and r_2 are their respective magnitudes and $\theta_1$and θ_2 are their respective angles, then their product is $z_1 z_2 = r_1 e^{i\theta_1} r_2 e^{i\theta_2} = r_1 r_2 e^{i\theta_1 + i\theta_2} = r_1 r_2 e^{i(\theta_1 + \theta_2)}$, and this last expression implies that the magnitude of the product is $r_1 r_2$ and the angle of the product is $\theta_1 + \theta_2 \; modulo \; 360°$. Hence, the theorem is verified.

□

Comment: We express our final angles $modulo \; 360°$ only for convenience, and the above theorem is true even without this reduction. Additionally, notice that since every complex number on the unit circle has magnitude equal to 1, it follows that the product of any two complex numbers on the unit circle will be another complex number on the unit circle whose angle will be the sum of the two angles of the two numbers that we will normally express $modulo \; 360°$. Furthermore, when convenient, we can also express angles by their corresponding complex numbers on the unit circle, and keep in mind that the n^{th} harmonic angles of astrology are exactly the same as the n^{th} roots of unity angles used in complex analysis. Additionally, recall that in each case we will refer to the smallest such angle θ with $0° \leq \theta < 360°$ as the *fundamental* or *root harmonic* for that given value of *n.* Now let's look at some more theorems!

Theorem: If θ is an n^{th} harmonic angle equal to $k\left(\frac{360^{\circ\circ}}{n}\right)$ for a natural number n where $k = 0,1,2,\ldots, n-1$, then any integer multiple of θ $modulo$ $360°$ is also an n^{th} harmonic angle.

Proof: This follows immediately from the definition of a n^{th} harmonic angle.

□

Corollary: If θ_1 and θ_2 are angles such that $\theta_2 - \theta_1$ is an n^{th} harmonic angle and if k is a natural number, then the difference between the corresponding angles in the k^{th} harmonic is also an n^{th} harmonic angle.

Proof: Since natural number (or even integer) multiples of an n^{th} harmonic angle result in an n^{th} harmonic angle, it follows that $k\theta_2 - k\theta_1 \; modulo \; 360° = k(\theta_2 - \theta_1) \; modulo \; 360°$ is an n^{th} harmonic angle.

□

Comment: One of the things the above result shows us is that planets do not wind up just anywhere in an n^{th} harmonic chart. Some structure of the original birth chart is always preserved. Thus, if two planets are $90°$ apart in a birth chart, then in any n^{th} harmonic chart for either an integer or a natural number n, the difference between the transformed angles will always be an n^{th} harmonic angle. That is, either $0°, 90°, 180°$, or $270°$. Hence, if we think of the 4th harmonic in astrology as involving tension, then some form of this tension will persist in each of the various natural number harmonic charts. Below, for example, are the values in various n^{th} harmonics for the 4th harmonic angle $90°$.

HARMONIC	NTH HARMONIC *MODULO* 360
1	90
2	180
3	270
4	0
5	90
6	180
7	270
8	0

Theorem: If θ is an n^{th} harmonic angle for some natural number n, then $n\theta = 0°$ $(modulo \; 360°)$. In other words, θ is equivalent to $0°$ in an n^{th} harmonic chart.

Proof: If θ is an n^{th} harmonic angle for a natural number n, then $\theta = k\left(\frac{360^{\circ\circ}}{n}\right)$ where *k* is an integer. Thus, $n\theta = n \times \; k\left(\frac{360^{\circ\circ}}{n}\right) = k(360°)$ which implies that $n\theta = 0° \; modulo \; 360°$ and that θ is equivalent to $0°$ in an n^{th} harmonic chart. □

Comment: This is a simple, but very useful result as it lets us know all n^{th} harmonic that all n^{th} harmonic angles will be equivalent to 0° in an n^{th} harmonic chart.

Theorem: If θ is equivalent to 0° in an n^{th} harmonic chart, then θ is an n^{th} harmonic angle.

Proof: The angle that θ gets mapped to in the n^{th} harmonic chart is $n\theta\ modulo\ 360°$, and $n\theta = 0\ modulo\ 360°$ if and only if $n\theta = k(360°)$ for some integer k. But this means that $\theta = k\left(\frac{360°}{n}\right)$, and, hence, θ is an n^{th} harmonic angle.

□

Comment: Again, it is very useful to know that the only angles that will be equivalent to 0° in an n^{th} harmonic chart are precisely those that are n^{th} harmonic angles.

Theorem: The angle 0° is equivalent to 0° in all n^{th} harmonic charts.

Proof: This is obvious since $n(0°) = 0°\ modulo\ 360°$.

□

Theorem: If θ is an n^{th} harmonic angle, then $(n+1)\theta = \theta\ modulo\ 360°$, or, in other words, θ is a fixed point in an $(n+1)^{th}$ harmonic chart.

Proof: If θ is an n^{th} harmonic angle, then $\theta = k\left(\frac{360^{\circ\circ}}{n}\right)$ where $k = 0,1,2,\ldots,n-1$. Thus, $(n+1)\theta = (n+1)k\left(\frac{360^{\circ\circ}}{n}\right) = k(360°) + k\left(\frac{360^{\circ\circ}}{n}\right) = k\left(\frac{360^{\circ\circ}}{n}\right)\ modulo\ 360° = \theta\ modulo\ 360°$.

□

Comment: In mathematics, points that remain fixed under a given transformation are always of interest, and once again this shows us that harmonic charts preserve more of the structure of the original chart than first realized.

Theorem: If θ is a fixed point in an $(n+1)^{th}$ harmonic chart, then θ is an n^{th} harmonic angle.

Proof: If θ is a fixed point in an $(n+1)^{th}$ harmonic chart, then $(n+1)\theta = \theta\ modulo\ 360°$ implies that $n\theta + \theta = \theta\ modulo\ 360°$ which implies that $n\theta = 0°\ modulo\ 360°$. But this, in turn, means that $n\theta = k(360°)$ for some integer k, and, hence, $\theta = k\left(\frac{360°}{n}\right)$, and, therefore, θ is an n^{th} harmonic angle.

□

Comment: Just as we showed that the only angles that go to $0°$ in an n^{th} harmonic chart, so is it the case that the only angles that are fixed when we transform to an $(n+1)^{th}$ harmonic chart are precisely those angles in the original chart that are n^{th} harmonic angles.

Theorem: If n and m are natural numbers and if n is a factor of m, then each of the n^{th} harmonic angles is equivalent to $0°\ modulo\ 360°$ in the m^{th} harmonic chart.

Proof: Suppose that $\theta = k\left(\frac{360°}{n}\right)$ is an n^{th} harmonic angle and that $m = qn$ where $q, n \in \mathbb{N}$. Then the m^{th} harmonic of θ is $m\theta = qn\theta = qnk\left(\frac{360°}{n}\right) = qk(360°) = 0°\ modulo\ 360°$. Therefore, each of the n^{th} harmonic angles is equivalent to $0°\ modulo\ 360°$ in the m^{th} harmonic chart.

□

Comment: Again, this is a useful result that can allow you to arrive at some results automatically.

Theorem: If m is equal to a natural number power of $n+1$, then each of the n^{th} harmonic angles is an unchanged fixed point when the original chart is transformed into the m^{th} harmonic.

PROOF: If θ is an n^{th} harmonic angle, then $\theta = k\left(\frac{360^{\circ\circ}}{n}\right)$ where k is an integer. Thus, $m\theta = (n+1)^j\theta = (n+1)(n+1)\ldots(n+1)k\left(\frac{360^{\circ\circ}}{n}\right)$ for some natural number j. Since by previous proof we know that $(n+1)\theta = (n+1)k\left(\frac{360^{\circ\circ}}{n}\right) = nk\left(\frac{360^{\circ\circ}}{n}\right) + k\left(\frac{360^{\circ\circ}}{n}\right) = nk\left(\frac{360^{\circ\circ}}{n}\right) + \theta = \theta\ modulo\ 360°$, it follows that every time we multiply $k\left(\frac{360^{\circ\circ}}{n}\right)$ by another factor of $(n+1)$, the result is always equivalent to $\theta\ modulo\ 360°$. Hence, it follows that $(n+1)^j\theta = \theta\ modulo\ 360°$, and, thus, θ is a fixed point in the m^{th} harmonic chart.

□

Comment: Thus, not only are n^{th} harmonic angles fixed in the $n+1$ harmonic chart, they're also fixed in any $(n+1)^j$ harmonic chart where j is a natural number.

Theorem: If m is not a power of $n+1$ and if n is not a factor of m, then the m^{th} harmonic will produce a permutation of the n^{th} harmonic angles.

Proof: Suppose that $\theta = k\left(\frac{360°}{n}\right)$, for some natural number k with $0 \leq k < n$, is an n^{th} harmonic angle and that m is not a power of $n+1$ and n is not a factor of m. Then $m\theta = mk\left(\frac{360°}{n}\right)\ modulo\ 360°$ is also an n^{th} harmonic angle since m is not divisible by n. Now suppose that there exist integers k_1, k_2 such that $k_1 \neq k_2, k_1 \& k_2 \in \{0,1,\ldots,n-1\}$, and $mk_1\left(\frac{360°}{n}\right) = mk_2\left(\frac{360°}{n}\right)\ modulo\ 360°$. Then $m(k_1 -$

$k_2)\left(\frac{360°}{n}\right) = 0°\ modulo\ 360°$ which implies that n divides $k_1 - k_2$. However since $k_1 - k_2 \leq n - 1$, this is not possible. Therefore, the m^{th} harmonic produces a permutation of the n^{th} harmonic angles.

$\square$

Comment: Once more we see that harmonic charts possess incredible structure and that n^{th} harmonic angles remain n^{th} harmonic angles in higher, natural number harmonic charts.

Theorem: If θ is a root m^{th} harmonic angle for some natural number m and a natural number n divides m, then the root n^{th} harmonic angle is a natural number multiple of the root m^{th} harmonic angle.

Proof: If θ is an m^{th} harmonic angle, then the root angle is $\frac{360°}{m}$. But if $m = fn$ for some $f \in \mathbb{N}$, then $\frac{360°}{m} = \frac{360°}{fn} = \frac{1}{f}\left(\frac{360°}{n}\right)$. Hence, the n^{th} harmonic root angle is $\frac{360°}{n} = f\left(\frac{360°}{fn}\right) = f\left(\frac{360°}{m}\right) = f\theta$. Therefore, the root n^{th} harmonic angle is a natural number multiple of the root m^{th} harmonic angle.

$\square$

Comment: Again, this is another potentially useful result that is good to know as we continue to complete the mathematical theory of harmonic charts.

Theorem: If an angle θ can be written as $\theta = \theta_1 + \theta_2$ where θ_1, θ_2 are n^{th} and m^{th} harmonic angles for natural numbers n and m, respectively, and if k is an integer, then $k\theta$ can also be written as the sum of n^{th} and m^{th} harmonic angles.

Proof: This follows immediately from the fact that $k\theta = k(\theta_1 + \theta_2) = k\theta_1 + k\theta_2$, and by our previous results the latter is sum of n^{th} and m^{th} harmonic angles.

$\square$

Comment: This is a very interesting result, so let's consider the ramifications by taking a simple example. Hence, consider $210° = 120° + 90°$, the sum of a 3rd harmonic angle with a 4th harmonic angle. If we now look at various natural number harmonics (as indicated in the table below), then we can experience for ourselves that the harmonic of a 3rd harmonic angle is another 3rd harmonic angle, the harmonic of a 4th harmonic angle is another 4th harmonic angle, and the sum of these new harmonic angles is equivalent $modulo$ 360° to the new harmonic of 210°.

HARMONIC	NTH HARMONIC *MODULO* 360	THETA1	THETA2	THETA1 + THETA 2 *MODULO* 360
1	210	120	90	210
2	60	240	180	60
3	270	0	270	270
4	120	120	0	120
5	330	240	90	330
6	180	0	180	180

Theorem: If $z = re^{i\theta}$ and $\theta = \theta_1 + \theta_2$ where θ_1, θ_2 are n^{th} and m^{th} harmonic angles for natural numbers n and m, and if k is an integer, then z^k can be written as a product of complex numbers with angles that are n^{th} and m^{th} harmonic angles.

Proof: We prove the result for $z = re^{i\theta}$, but it applies in particular to complex numbers on the unit circle which always have the form $z = e^{i\theta}$. Thus, if $z = re^{i\theta}$ and $\theta = \theta_1 + \theta_2$ where θ_1, θ_2 are n^{th} and m^{th} harmonic angles and if k is an integer, then $z^k = \left(re^{i\theta}\right)^k = r^k e^{ik\theta} = r^k e^{ik(\theta_1+\theta_2)} = \left(r^k e^{ik\theta_1}\right)e^{ik\ \ {}_2}$, and by our previous results, $k\theta_1$ and $k\theta_2$ are n^{th} and m^{th} harmonic angles, respectively.

□

Comment: Once more we see that a lot of our original structure is preserved in harmonic charts.

Theorem: If $\theta = \theta_1 + \theta_2$ where θ_1, θ_2 are n^{th} and m^{th} harmonic angles for natural numbers n and m and if $q = lcm(n, m)$ is the least common multiple of m and n, then θ is a q harmonic angle.

Proof: Since θ_1 is an n^{th} harmonic angle and since q is a multiple of n, it follows that $q\theta_1 = 0°\ modulo\ 360°$. And similarly, that $q\theta_2 = 0°\ modulo\ 360°$. Hence, $q\theta = q\theta_1 + q\theta_2 = 0°\ modulo\ 360°$, and, thus, θ is a q harmonic angle. Furthermore, since $q = lcm(n, m)$, it is the smallest such natural number.

□

Commentary: I find this result both quite interesting and quite useful as it tells us, for example, that the sum of a 2^{nd} harmonic angle and a 3^{rd} harmonic angle will always be a 6^{th} harmonic angle. Thus, for instance, $180° + 120° = 300°$ is a 6^{th} harmonic angle since $300° = 5 \cdot 60° = 5 \cdot \frac{360°}{6}$.

Corollary: If an angle is the sum of n^{th} and m^{th} harmonic angles for some natural numbers n and m with θ_1 and θ_2 being the respective n^{th} and m^{th} root harmonic angles and if $q = lcm(n, m)$ is the least common multiple of m and n, then $\theta = \frac{360°}{q}$, the root q^{th} harmonic angle, is the greatest common divisor of θ_1 and θ_2. In symbols, $\theta = gcd(\theta_1, \theta_2)$.

<u>Proof</u>: Suppose θ_1, θ_2 are the respective n^{th} and m^{th} root harmonic angles for some natural numbers n and m, and suppose also that $q = lcm(n, m)$ and $\theta = \frac{360°}{q}$. Then $\theta_1 = \frac{360°}{n}$ and $\theta_2 = \frac{360°}{m}$, and $\theta = \frac{360°}{q}$ is a root q^{th} harmonic angle. Since $q = lcm(n, m)$, we can write $q = nq_1 = mq_2$. Thus, $\theta = \frac{360°}{q} = \frac{360°}{nq_1} = \frac{360°/n}{q_1} = \frac{\theta_1}{q_1} \Rightarrow \theta_1 = q_1 \cdot \theta$ and $\theta = \frac{360°}{q} = \frac{360°}{mq_2} = \frac{360°/m}{q_2} = \frac{\theta_2}{q_2} \Rightarrow \theta_2 = q_2 \cdot \theta$. Hence, θ_1 and θ_2 are both natural number multiples of θ, and, thus, θ is a common divisor of θ_1 and θ_2. Our claim now is that θ is the greatest common divisor of θ_1 and θ_2, and to show this let's assume that there exists another common divisor θ' such that $\theta' > \theta$. Then since θ' is a divisor of θ_1 and θ_2, there exist $q_1{}'$ and $q_2{}'$ such that $\theta_1 = q_1{}'\,\theta'$ and $\theta_2 = q_2{}'\theta$. Hence, $\theta' = \frac{\theta_1}{q_1{}'} = \frac{360°/n}{q_1'} = \frac{360°}{nq_1{}'}$ and $\theta' = \frac{\theta_2}{q_2{}'} = \frac{360°/m}{q_2{}'} = \frac{360°}{mq_2{}'}$. From this it follows that $nq_1{}' = mq_2{}'$, and if we set $q' = nq_1{}' = mq_2{}'$, then $\theta' = \frac{360°}{q'}$. From this we can conclude that q' is a common multiple of n and m and that $q' = \frac{360°}{\theta'}$. However, since we assumed that $\theta' > \theta$, it follows that $q' = \frac{360°}{\theta'} < \frac{360°}{\theta} = q$. But since q' is a common multiple of n and m, this contradicts our assertion that $q = lcm(n, m)$, and the assumption that led to this contradiction was that θ was not the greatest common divisor of θ_1 and θ_2. Therefore, $\theta = gcd(\theta_1, \theta_2)$.

□

<u>Comment</u>: The angle $210°$ is equal to $90° + 120°$, a 4th harmonic angle plus a 3rd harmonic angle. By our theorem, it should follow that $210°$ is a 12th harmonic angle and that $\frac{360°}{12} = 30°$ is the greatest common divisor for 90° and 120°, and that is indeed the case.

<u>Theorem</u>: If an angle is the sum of two angles θ_1 and θ_2 where θ_1, θ_2 are n^{th} and m^{th} harmonic angles, respectively, for natural numbers n and m and if $q = lcm(n, m)$ and $\theta = gcd(\theta_1, \theta_2)$, then $\frac{360°}{q} = \theta = \theta_1 + \theta_2$ and $n\theta$ is an m^{th} harmonic angle.

<u>Proof</u>: It follows from previous proofs that $\frac{360°}{q} = \theta = \theta_1 + \theta_2$, and since where θ_1, θ_2 are n^{th} and m^{th} harmonic angles, respectively, for natural numbers n and m, we have that $n\theta = n\theta_1 + n\theta_2 = 0° + n\theta_2 \ modulo\ 360° = n\theta_2 \ modulo\ 360°$, and since θ_2 is an m^{th} harmonic angle, it follows that $n\theta_2$ is also an m^{th} harmonic angle.

□

<u>Comment</u>: Nice!

<u>Theorem</u>: If m and n are natural numbers with no common divisor other than 1 and if $q = lcm(m, n) = mn$ and $\theta = \frac{360°}{q}$, then $n\theta$ is an m^{th} harmonic angle and $m\theta$ is an n^{th} harmonic angle.

Proof: Since θ is a q harmonic angle, it follows from previous proof that $q\theta = 0^\circ\ modulo\ 360^\circ$. However, since $q = mn$, it follows that $0^\circ\ modulo\ 360^\circ = q\theta = m(n\theta) = n(m\theta)$. These last two inequalities imply that $n\theta$ is an m^{th} harmonic angle and $m\theta$ is an n^{th} harmonic angle.

□

Corollary: If m and n are natural numbers with no common divisor other than 1 and if $q = lcm(m, n) = mn$ and j, k are natural numbers such that $j = kq = kmn$ and if $\theta = \frac{360^\circ}{j}$, then $kn\theta$ is an m^{th} harmonic angle and $km\theta$ is an n^{th} harmonic angle.

Proof: Since $0^\circ\ modulo\ 360^\circ = j\theta = kq\theta = m(kn\theta) = n(km\theta)$, it follows that $kn\theta$ is an m^{th} harmonic angle and $km\theta$ is an n^{th} harmonic angle.

□

Comment: The better you know and understand these theorems, the more you will be able to play harmonic charts like a violin, and these theorems are also stepping stones to new theorems yet to come!

Theorem: If an angle θ as measured in degrees is a rational fraction of 360° of the form $\frac{360^\circ}{f/g} = \frac{g(360^\circ)}{f}$, where $f, g \in \mathbb{N}$, then $f\theta = 0^\circ\ modulo\ 360^\circ$ and $(f + 1)\theta = \theta\ modulo\ 360^\circ$.

Proof: Clearly, $f\theta = f\left(\frac{g(360^\circ)}{f}\right) = g(360^\circ) = 0^\circ\ modulo\ 360^\circ$, and $(f + 1)\theta = (f + 1)\left(\frac{g(360^\circ)}{f}\right) = g(360^\circ) + \frac{g(360^\circ)}{f} = 0^\circ + \theta = \theta\ modulo\ 360^\circ$.

□

Comment: This theorem helps to extend out results from natural number harmonics to rational number harmonics.

Theorem: If $\theta = \frac{360^\circ}{f}$ where f is irrational, then θ is irrational, and there is no natural number g such that $g\theta = 0^\circ\ modulo\ 360^\circ$.

Proof: If θ is rational and f is irrational, then that would mean that $f\theta = 360^\circ$ is irrational. Clearly not true. Therefore, θ is irrational. Thus, it also follows that $g\theta$ is irrational and never equal to a natural number multiple of 360°. Hence, there is no natural number g such that $g\theta = 0^\circ\ modulo\ 360^\circ$.

□

Corollary: If $\theta = \frac{360^\circ}{f}$ where f is irrational, then there is no natural number $g > 1$ such that $g\theta = \theta\ modulo\ 360^\circ$.

Proof: If there were a natural number $g > 1$ such that $g\theta = \theta\ modulo\ 360°$, then we would have $(g-1)\theta = g\theta - \theta = \theta - \theta = 0°\ modulo\ 360°$ in contradiction to our theorem.

□

Commentary: As noted in our chapter on *Astrology and Fractals*, irrational number harmonics could lead to *strange attractors*. However, we also note that every irrational number can be approximated by a rational number. Thus, the rational numbers may be sufficient for all practical applications.

Theorem: Let $\theta = k\left(\frac{360°}{n}\right)$ be a nonzero n^{th} harmonic angle, and let m be a natural number. If $\theta = k\left(\frac{360°}{n}\right)$ is fixed in the m^{th} harmonic, then all n^{th} harmonic angles are fixed in the m^{th} harmonic.

Proof: Let $\theta = k\left(\frac{360°}{n}\right)$ be a nonzero n^{th} harmonic angle for some natural number k, and suppose m is a natural number such that $m\theta = \theta\ modulo\ 360°$. Then $mk\left(\frac{360°}{n}\right) = k\left(\frac{360°}{n}\right)\ modulo\ 360°$, and this implies via division by k that $m\left(\frac{360°}{n}\right) = \frac{360°}{n}\ modulo\ 360°$. Hence, it now follows that for any natural number f that $mf\left(\frac{360°}{n}\right) = fm\left(\frac{360°}{n}\right) = f\left(\frac{360°}{n}\right)\ modulo\ 360°$. Therefore, if one nonzero n^{th} harmonic angle is fixed in the m^{th} harmonic, then all n^{th} harmonic angles are fixed in the m^{th} harmonic.

□

Comment: This strengthens one of our earlier results.

Theorem: The n^{th} roots of unity form a finite cyclic group of order n under addition that is isomorphic to $\mathbb{Z}_n$.

Proof-1: Obvious.

Proof-2: Easy.

Proof-3: Clear.

Proof-4: Why do you want more proofs? I've already given you three!! 😊

□

Comment: If you want something done right, do it yourself!