ANOTHER APPLICATION

Discussion: Frankly, I found this result rather interesting!

Theorem: A group of permutations of odd order consists of only even permutations.

Proof: Let G be a group of permutations such that $|G|$ is odd. If $|G| = 1$, then the only permutation in G is the identity which is even. If $|G| = 3$, then G is the cyclic group of order 3 (since up to isomorphism there exists only one group of order 3), we can represent the permutations as $\{(\ ), (1,2,3), (1,3,2)\}$, and each of these permutations is even. Thus, assume that $|G|$ is odd and greater than three. Then G has more than two elements which are not the identity. Also, as before, define $f : G \to \mathbb{Z}_2$ by

$$f(p) = \begin{cases} 0 & \text{if } p \text{ is an even permutation} \\ 1 & \text{if } p \text{ is an odd permutation} \end{cases}.$$

If $Ker(f) = G$, then every permutation in G is even, and we are done. Thus, assume that not every permutation in G is even, i.e. that some are odd. In this case, $f : G \to \mathbb{Z}_2$ will be an onto function, and we have that $2 = |\mathbb{Z}_2| = |G/Ker(f)| = |G|/|Ker(f)| \Rightarrow |G| = 2 \cdot Ker(f)$. But this contradicts our assumption that $|G|$ is odd, and therefore, a group of permutations of odd order consists of only even permutations.

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