AN APPLICATION

Discussion: Recall that in a permutation group, every permutation can be classified as even or odd and that in Theorem 29 we showed that the even permutations form a normal subgroup of our permutation group. In particular, if S_n is the group of all permutations that can be made of *n* objects, then the normal subgroup of all even permutations is called the alternating group of degree *n*, A_n, and in the theorem below we show that for $n \geq 2$, the number of elements in A_n is $|A_n| = \dfrac{n!}{2}$.

Definition: If S_n is the group of all permutations that can be made of *n* objects (known as the symmetric group of degree *n*), then the alternating group of degree *n*, A_n, is the subgroup of all even permutations in S_n. Also, since the identity is counted as an even permutation, this subgroup of S_n always exists.

Theorem: If S_n is the symmetric group of degree *n* for $n \geq 2$, then $|A_n| = \dfrac{|S_n|}{2} = \dfrac{n!}{2}$.

Proof: We can define a surjective (onto) homomorphism $f : S_n \to \mathbb{Z}_2$ by

$$f(p) = \begin{cases} 0 & \text{if } p \text{ is an even permutation} \\ 1 & \text{if } p \text{ is an odd permutation} \end{cases}.$$

For S_n with $n \geq 2$, it should be clear that S_n will contain both even and odd permutations. For example, it contains the identity which is an even permutation, and it contains transpositions of two elements which are odd permutations. Thus, $S_n \neq A_n$. However, A_n is the *Kernel of f*, and, thus, $\mathbb{Z}_2 \cong S_n / Ker(f) = S_n / A_n$. From this it immediately follows that $2 = |\mathbb{Z}_2| = |S_n / A_n| = |S_n| / |A_n| = \dfrac{n!}{|A_2|} \Rightarrow |A_n| = \dfrac{n!}{2}$.

$\square$