A FOURTH RESULT

Discussion: We introduced the notion of a commutator back in our section on beginning group theory, and in Theorem 21 we proved that the subgroup generated by forming all finite products of the commutators in our group is a normal subgroup of our group. In the theorem below, we show that the quotient group of our group by the commutator subgroup is always abelian. Furthermore, it could even be shown that the Kernel of any quotient group that is abelian must contain this commutator subgroup. However, this last part we leave for you to ponder.

Theorem: Let G be a group, and let G' be the derived or commutator subgroup, the subgroup generated by all products in G of the form $a^{-1}b^{-1}ab$. Then G/G' is abelian.

Proof: Let $\pi : G \to G/G'$ be the natural map where $\pi(g) = G'g$. To show that G/G' is abelian, we need to show that if $G'a, G'b \in G/G'$, then $G'aG'b = G'bG'a$. Another way to express this equation is as $(G'a)^{-1}(G'b)^{-1}G'aG'b = G'$, the identity in G/G'. However, this is easy to verify since $(G'a)^{-1}(G'b)^{-1}G'aG'b = (G'a^{-1})(G'b^{-1})G'aG'b = G'(a^{-1}b^{-1}ab)$. Hence, since $a^{-1}b^{-1}ab$ is a commutator, it follows that $G'(a^{-1}b^{-1}ab) = G'$, the identity in G/G'. Therefore, $(G'a)^{-1}(G'b)^{-1}G'aG'b = G'(a^{-1}b^{-1}ab) = G'$ implies that $G'aG'b = G'bG'a$ and G/G' is abelian.

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