A CHILD'S GARDEN OF GROUPS

Visual Representations of Groups

(Part 5)

by

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INTRODUCTION (PART 5)

In part 5 of this work we introduce three different ways to create visual representations of *groups* – a *cycle graph*, a *Cayley diagram* (named after mathematician Arthur Cayley, 1821-1895), and last but not least, what I like to call a *generator diagram*. These three methods become more complex as the size of our *group* increases, but for relatively small *groups*, they provide an interesting way to study the subject, and these are methods that generally aren't covered much in traditional textbooks on *group theory* and *abstract algebra*. Additionally, toward the end of Part 5 we have another chapter of *How to Use GAP*. Nothing new is given that is not also given in the previous Part 4, but that chapter from Part 4 is repeated here for reference.

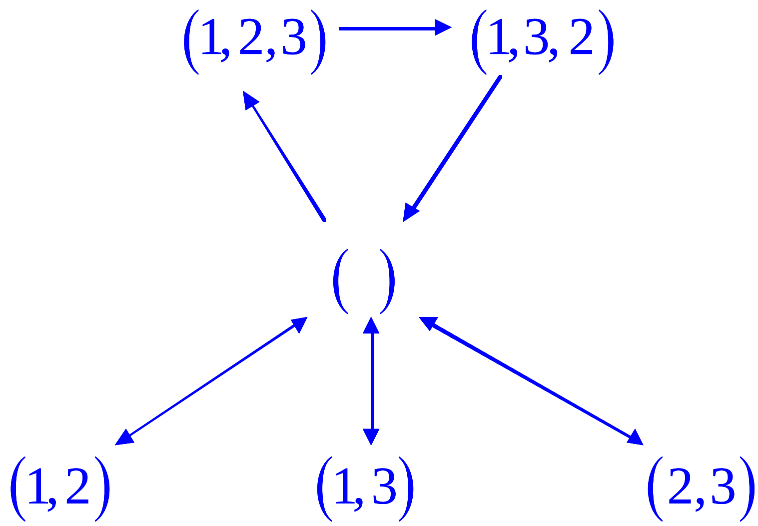
VISUAL REPRESENTATIONS OF GROUPS

There are three ways of representing *groups* visually that I want to talk about. I call these ways *cycle graphs*, *Cayley diagrams*, and *generator diagrams*. Of these three, only *Cayley diagrams* were easy to find information on when I was in graduate school in the early eighties. *Cycle graphs* may have originated or been popularized later. And lastly, *generator diagrams* are a creation of my own even though I wouldn't be completely surprised if I am not the first one to ever use such a diagram. We'll illustrate each of these displays by first applying them to the *dihedral group* $D_3 \cong S_3$, the *group* of symmetries of an equilateral triangle.

Let's begin with the list of elements in D_3.

$$\left\{ \begin{matrix} (\ ) \\ (1,2) \\ (1,3) \\ (2,3) \\ (1,2,3) \\ (1,3,2) \end{matrix} \right\} \cong D_3$$

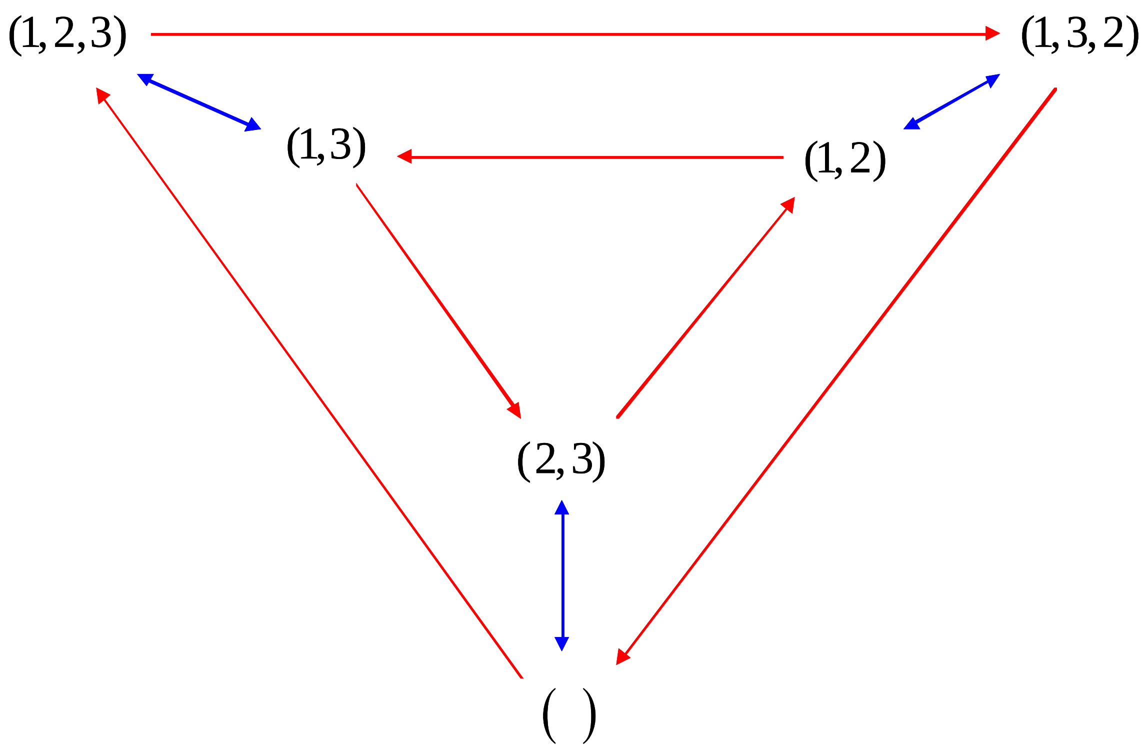
From this representation we can see that D_3 contains one element of order 1, three elements of order 2, and two elements of order 3. A *cycle graph* for this simply creates a visual display of the order of each element. I draw mine as follows.



In a *cycle graph*, every element of the *group* should be contained in one of the cycles shown. Also, it would be nice if every single *group* was distinguished by the lengths of the cycles of its elements. Unfortunately, this is not always the case. For example, $C_4 \times C_2 \times C_2$ and $(C_4 \times C_2) \rtimes C_2$ are both *groups* of order 16, and they both have one element of order 1, seven elements of order 2, and eight elements of order 4. However, in spite of this similarity, the two *groups* are not *isomorphic*. However, for *groups* of order less than 16 the cycle structure is unique. For example, the *dihedral group* D_3 has one element of order 1, three elements of order 2, and two elements of order 3, and any other *group* of order 6 which has this same cycle structure must automatically be *isomorphic* to D_3. But again, why this does not hold true for $C_4 \times C_2 \times C_2$ and $(C_4 \times C_2) \rtimes C_2$ is something that is worthy of lengthy contemplation.

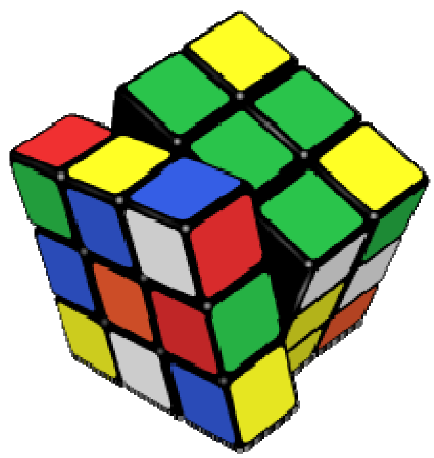
Cayley diagrams were invented by British mathematician Arthur Cayley (1821-1895). There exist some variations of his method for presenting *groups* visually, but the way I construct them is very easy if the elements of your *group* are expressed in terms of permutations. For example, if we want to create a *Cayley diagram* for the *dihedral group* D_3 (which is *isomorphic* to the *symmetric group*

S_3), then all we have to do is pick a minimal set of elements which generate the entire group such as $(1,2,3)$ and $(2,3)$. Next we create a diagram which shows the result of multiplying various combinations of those permutations together, and we represent multiplication by $(1,2,3)$ by an arrow of one color and multiplication by $(2,3)$ by an arrow of a different color. We begin our multiplication at the identity element $(\ )$, and notice that each element in the *group* will have two arrows leaving it, one for multiplication by $(1,2,3)$ and the other by multiplication by $(2,3)$. Below is the final result.



Notice that if we start at the identity, then the blue arrow represents multiplication by $(2,3)$ and the red arrow leaving $(2,3)$ represents an additional multiplication by $(1,2,3)$. Hence, the *Cayley diagram* shows us visually that $(2,3)(1,2,3)=(1,2)$. Likewise, we can easily find the result of any other multiplication by simply following the correct arrows in sequence.

The third type of visual display that I like is what I call a *generator diagram*. It is inspired by the moves that generate the *Rubik's cube group*, and it is a classic example of a *group* of permutations that act upon a set of objects. In particular, recall that the entire *group* of permutations of the facelets of Rubik's cube can be generated by quarter turns of the right, left, up, down, front, and back faces of the cube. We denote these moves by *R, L, U, D, F,* and *B*, but if we write them in the order $BFUDLR$, then we can appropriately pronounce this "befuddler."



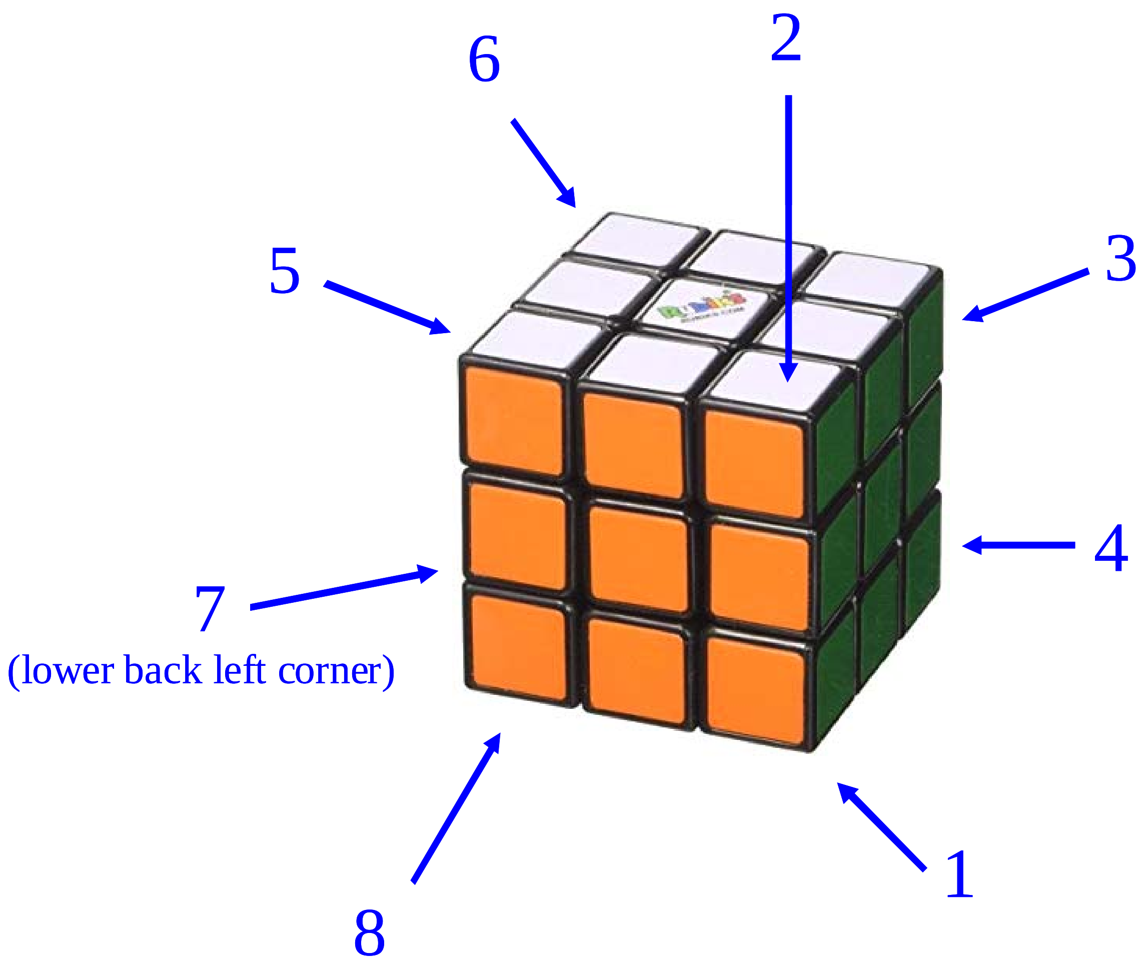
Recall also that if we number each facelet of our cube, then we can easily express the movements that generate our cube *group* as permutations written in cycle notation. The usual way for numbering the facelets is given below.

			1	2	3						
			4	UP	5						
			6	7	8						
9	10	11	17	18	19	25	26	27	33	34	35
12	LEFT	13	20	FRONT	21	28	RIGHT	29	36	BACK	37
14	15	16	22	23	24	30	31	32	38	39	40
			41	42	43						
			44	DOWN	45						
			46	47	48						

And now, using this labeling scheme, recall that we can express our generating moves as follows:

r =(25,27,32,30)(26,29,31,28)(3,38,43,19)(5,36,45,21)(8,33,48,24)
l =(9,11,16,14)(10,13,15,12)(1,17,41,40)(4,20,44,37)(6,22,46,35)
u =(1,3,8,6)(2,5,7,4)(9,33,25,17)(10,34,26,18)(11,35,27,19)
d =(41,43,48,46)(42,45,47,44)(14,22,30,38)(15,23,31,39)(16,24,32,40)
f =(17,19,24,22)(18,21,23,20)(6,25,43,16)(7,28,42,13)(8,30,41,11)
b=(33,35,40,38)(34,37,39,36)(3,9,46,32)(2,12,47,29)(1,14,48,27)

A simplified version of this, however, would be to ignore most of the color variations and just label the corner cublets 1 through 8.



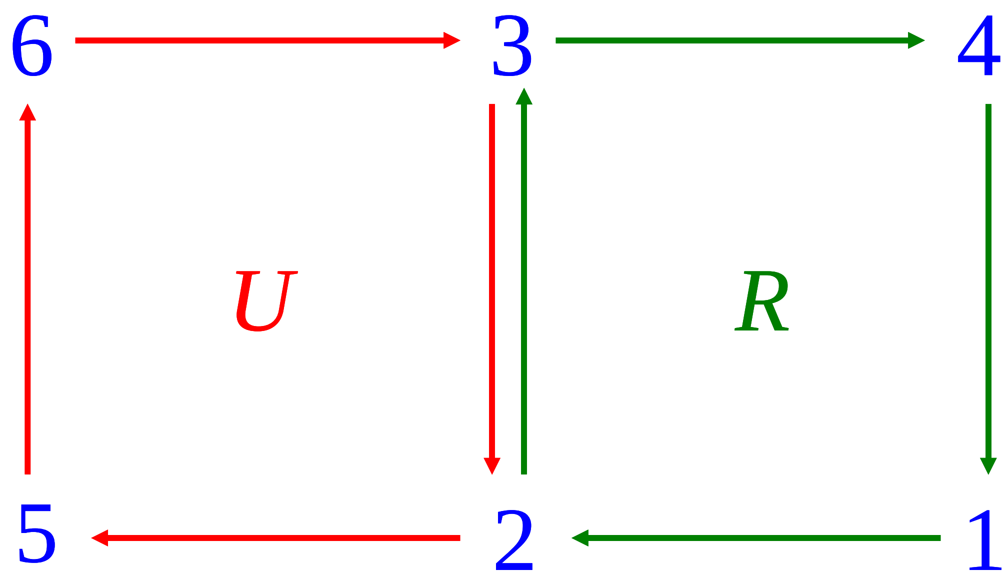
Using this simplification, we can rewrite the permutations for our moves as the following:

R = (1,2,3,4)

L = (5,8,7,6)

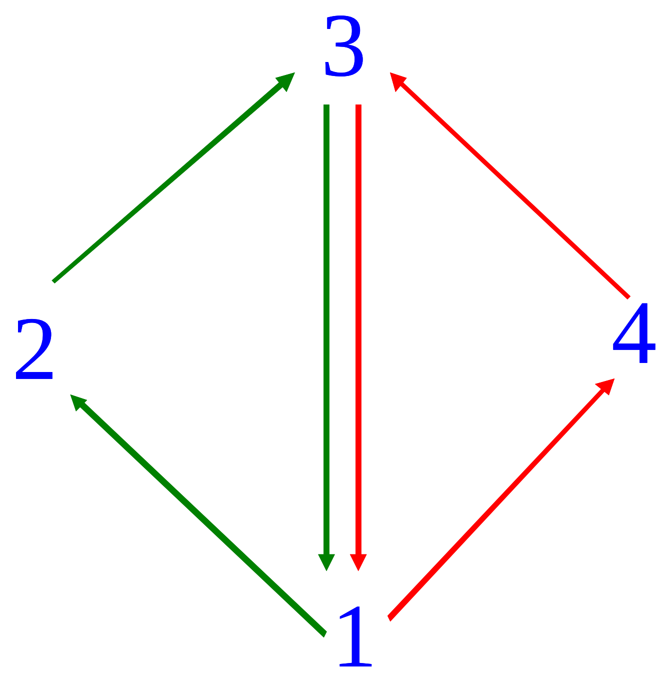
```
U = (2,5,6,3)
D = (1,4,7,8)
F = (1,8,5,2)
B = (4,3,6,7)
```

Of course, these permutations will generate something far smaller than the real *Rubik's cube group*, but, nonetheless, they will make it easier for us to explain *generator diagrams*. For instance, suppose we want to examine the *group* generated just by *R* and *U*. Then we can easily diagram this using a couple of permutations that act on the set of numbers $\{1,2,3,4,5,6\}$.

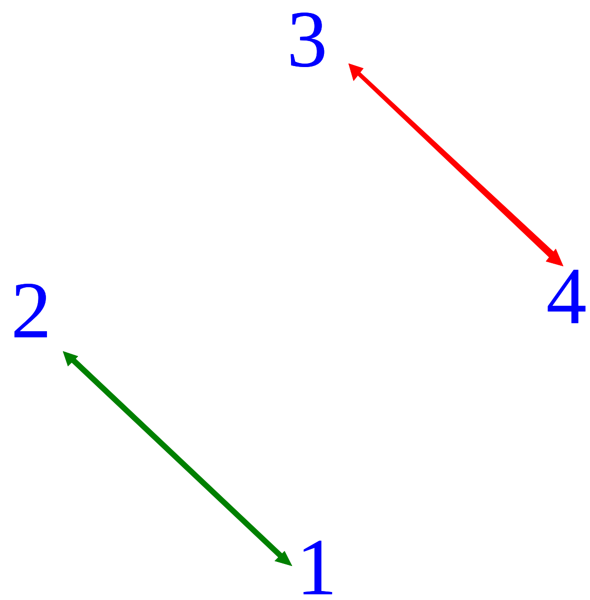


This diagram is what I call a *generator diagram*. It shows the objects that will be permuted along with the moves that create the corresponding *permutation group*. In this case, we generate our *group* by creating all possible finite combinations of our moves *R* and *U*, and the resulting *permutation group* is has 120 elements.

Below is another *generating diagram* that consists of the permutations $a=(1,2,3)$ and $b=(1,4,3)$. The resulting *group* that is generated has 12 elements and is *isopmorphic* to A_4, the *subgroup* of all even permutations contained in S_4 which, in turn, is the *group* of all possible permutations of a set of four elements.

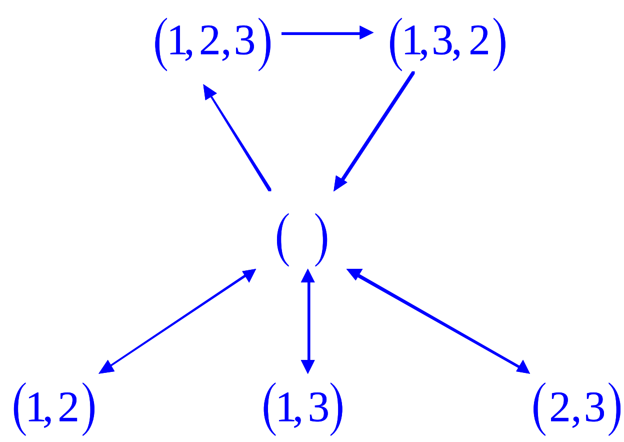


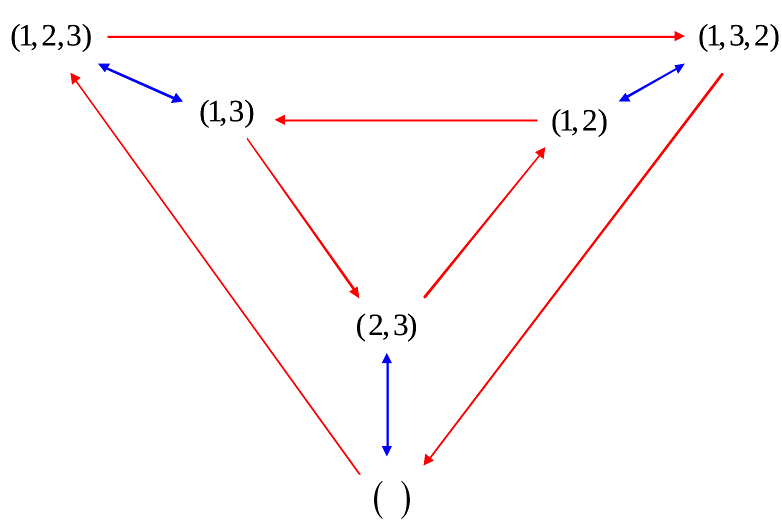
And now, here is a *generating diagram* for the *Klein 4-group*, $C_2 \times C_2$, where the generators are $a = (1,2)$ and $b = (3,4)$.



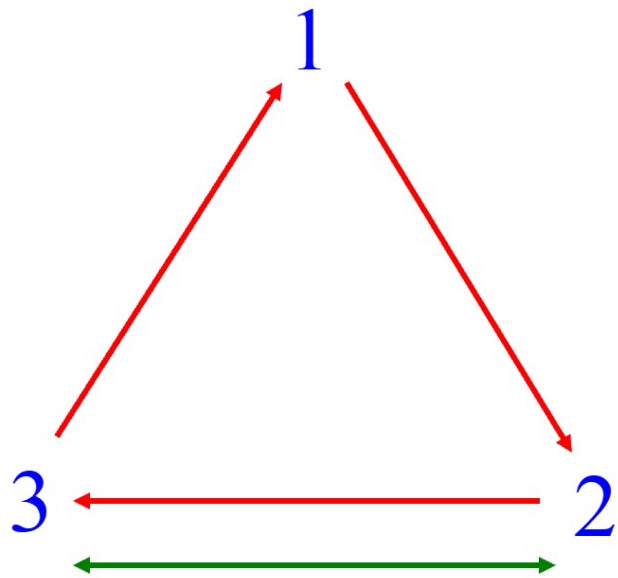
Am I the first person to ever represent a *group* in this way by using what I call *generator diagrams*? On the one hand, I rather doubt it, but on the other hand, I haven't really seen these diagrams used before, and, frankly, I find them very useful. I should also reiterate that when I was young and in graduate school back in the early eighties, I only ever saw *Cayley diagrams*, and they were not really covered in my classes as all the emphasis back in those days was placed on proving theorems.

Now let me show you something very interesting, but first recall what our *cycle* and *Cayley diagrams* look like for D_3, the symmetries of an equilateral triangle.





A *generator diagram* for D_3, in terms of permutations $a=(1,2,3)$ and $b=(2,3)$, can be constructed as follows.



However, since the *group* D_3 has six elements, a good question to ask is can we express our *generator diagram* in terms of permutations of the numbers of the set $\{1,2,3,4,5,6\}$? Fortunately, this can be done, and I'll now show you how to do it. We'll begin with the multiplication table for D_3.

*	()	(2,3)	(1,2)	(1,2,3)	(1,3,2)	(1,3)
()	()	(2,3)	(1,2)	(1,2,3)	(1,3,2)	(1,3)
(2,3)	(2,3)	()	(1,2,3)	(1,2)	(1,3)	(1,3,2)
(1,2)	(1,2)	(1,3,2)	()	(1,3)	(2,3)	(1,2,3)
(1,2,3)	(1,2,3)	(1,3)	(2,3)	(1,3,2)	()	(1,2)
(1,3,2)	(1,3,2)	(1,2)	(1,3)	()	(1,2,3)	(2,3)
(1,3)	(1,3)	(1,2,3)	(1,3,2)	(2,3)	(1,2)	()

And now, if you look closely at this table, it's easy to see that the elements in each row are just a permutation of the elements in the top row. That means that every element in the *group* can be represented by a permutation of the six elements in the top row, and the greater implication of this (known as Cayley's Theorem) is that every finite *group* is *isomorphic* to a *permutation group*. However, the permutation may not be what you think it is. One might be tempted to indicate the permutation associated with each element by drawing arrows as follows.

*	()	(2,3)	(1,2)	(1,2,3)	(1,3,2)	(1,3)
()	()	(2,3)	(1,2)	(1,2,3)	(1,3,2)	(1,3)
(2,3)	(2,3)	()	(1,2,3)	(1,2)	(1,3)	(1,3,2)
(1,2)	(1,2)	(1,3,2)	()	(1,3)	(2,3)	(1,2,3)
(1,2,3)	(1,2,3)	(1,3)	(2,3)	(1,3,2)	()	(1,2)
(1,3,2)	(1,3,2)	(1,2)	(1,3)	()	(1,2,3)	(2,3)
(1,3)	(1,3)	(1,2,3)	(1,3,2)	(2,3)	(1,2)	()

However, this is not going to work, and I'll show you why. First, notice that the red arrows above suggest that the permutation we want to associate this way with $(2,3)$ is $([(\ ),(2,3)][(1,2),(1,2,3)][(1,3,2)(1,3)])$. Now let's do the same thing for $(1,2,3)$ by using the table below.

*	()	(2,3)	(1,2)	(1,2,3)	(1,3,2)	(1,3)
()	()	(2,3)	(1,2)	(1,2,3)	(1,3,2)	(1,3)
(2,3)	(2,3)	()	(1,2,3)	(1,2)	(1,3)	(1,3,2)
(1,2)	(1,2)	(1,3,2)	()	(1,3)	(2,3)	(1,2,3)
(1,2,3)	(1,2,3)	(1,3)	(2,3)	(1,3,2)	()	(1,2)
(1,3,2)	(1,3,2)	(1,2)	(1,3)	()	(1,2,3)	(2,3)
(1,3)	(1,3)	(1,2,3)	(1,3,2)	(2,3)	(1,2)	()

Based on this correspondence, we should associate $(1,2,3)$ with the permutation $([(\ ),(1,2,3),(1,3,2)][(2,3),(1,3)(1,2)])$. Now if we do the multiplication $(2,3)(1,2,3)$ from left to right, then we get $(1,2)$, and using the table below we can see that $(1,2)$ corresponds to $([(\ ),(1,2)][(2,3),(1,3,2)][(1,2,3)(1,3)])$.

```
*        | ()      (2,3)   (1,2)   (1,2,3) (1,3,2) (1,3)
---------+----------------------------------------------
()       | ()      (2,3)   (1,2)   (1,2,3) (1,3,2) (1,3)
(2,3)    | (2,3)   ()      (1,2,3) (1,2)   (1,3)   (1,3,2)
(1,2)    | (1,2)   (1,3,2) ()      (1,3)   (2,3)   (1,2,3)
(1,2,3)  | (1,2,3) (1,3)   (2,3)   (1,3,2) ()      (1,2)
(1,3,2)  | (1,3,2) (1,2)   (1,3)   ()      (1,2,3) (2,3)
(1,3)    | (1,3)   (1,2,3) (1,3,2) (2,3)   (1,2)   ()
```

And now we can begin to see what the problem is. If we do the multiplication $\left([(\ ),(2,3)][(1,2),(1,2,3)][(1,3,2)(1,3)]\right)\left([(\ ),(1,2,3),(1,3,2)][(2,3),(1,3)(1,2)]\right)$ from left to right, then we get $\left([(\ ),(1,3)][(2,3),(1,2,3)][(1,2),(1,3,2)]\right)$. However, this is not the permutation that $(2,3)(1,2,3)=(1,2)$ corresponds to. In other words, our correspondence does not preserve the multiplication, and we call a one-to-one correspondence an *isomorphism* only if the multiplication in one *group* corresponds to the multiplication in the other, and the correspondence we set up doesn't do that! So how can we make things work the right way? Well, fortunately, it's not too hard. We just have to think in terms of permutations of positions! Thus, let's examine the table below.

Position

1 2 3 4 5 6

```
*        | ()      (2,3)   (1,2)   (1,2,3) (1,3,2) (1,3)
---------+----------------------------------------------
()       | ()      (2,3)   (1,2)   (1,2,3) (1,3,2) (1,3)
(2,3)    | (2,3)   ()      (1,2,3) (1,2)   (1,3)   (1,3,2)
(1,2)    | (1,2)   (1,3,2) ()      (1,3)   (2,3)   (1,2,3)
(1,2,3)  | (1,2,3) (1,3)   (2,3)   (1,3,2) ()      (1,2)
(1,3,2)  | (1,3,2) (1,2)   (1,3)   ()      (1,2,3) (2,3)
(1,3)    | (1,3)   (1,2,3) (1,3,2) (2,3)   (1,2)   ()
```

This table suggests that we want to construct the permutation corresponding to $(2,3)$ by noting where the contents of position 1 in the identity wind up in the row corresponding to $(2,3)$, and we can clearly see that the contents of position 1 are moved to position 2. If we do the same type of analysis for the remaining *group* elements, then we can conclude that the permutation corresponding to $(2,3)$ should be $(1,2)(3,4)(5,6)$. This means that the contents of positions 1 & 2 are switched as are the contents of positions 3 & 4 and the contents of positions 5 & 6. We can clearly see this in the table below.

Position

	1	2	3	4	5	6
*	()	(2,3)	(1,2)	(1,2,3)	(1,3,2)	(1,3)
()	()	(2,3)	(1,2)	(1,2,3)	(1,3,2)	(1,3)
(2,3)	(2,3)	()	(1,2,3)	(1,2)	(1,3)	(1,3,2)
(1,2)	(1,2)	(1,3,2)	()	(1,3)	(2,3)	(1,2,3)
(1,2,3)	(1,2,3)	(1,3)	(2,3)	(1,3,2)	()	(1,2)
(1,3,2)	(1,3,2)	(1,2)	(1,3)	()	(1,2,3)	(2,3)
(1,3)	(1,3)	(1,2,3)	(1,3,2)	(2,3)	(1,2)	()

Similarly, the element $(1,2,3)$ corresponds to $(1,5,4)(2,3,6)$.

Position

1 2 3 4 5 6

```
*        | ()      (2,3)   (1,2)   (1,2,3) (1,3,2) (1,3)
--------+------------------------------------------------
()       | ()      (2,3)   (1,2)   (1,2,3) (1,3,2) (1,3)
(2,3)    | (2,3)   ()      (1,2,3) (1,2)   (1,3)   (1,3,2)
(1,2)    | (1,2)   (1,3,2) ()      (1,3)   (2,3)   (1,2,3)
(1,2,3)  | (1,2,3) (1,3)   (2,3)   (1,3,2) ()      (1,2)
(1,3,2)  | (1,3,2) (1,2)   (1,3)   ()      (1,2,3) (2,3)
(1,3)    | (1,3)   (1,2,3) (1,3,2) (2,3)   (1,2)   ()
```

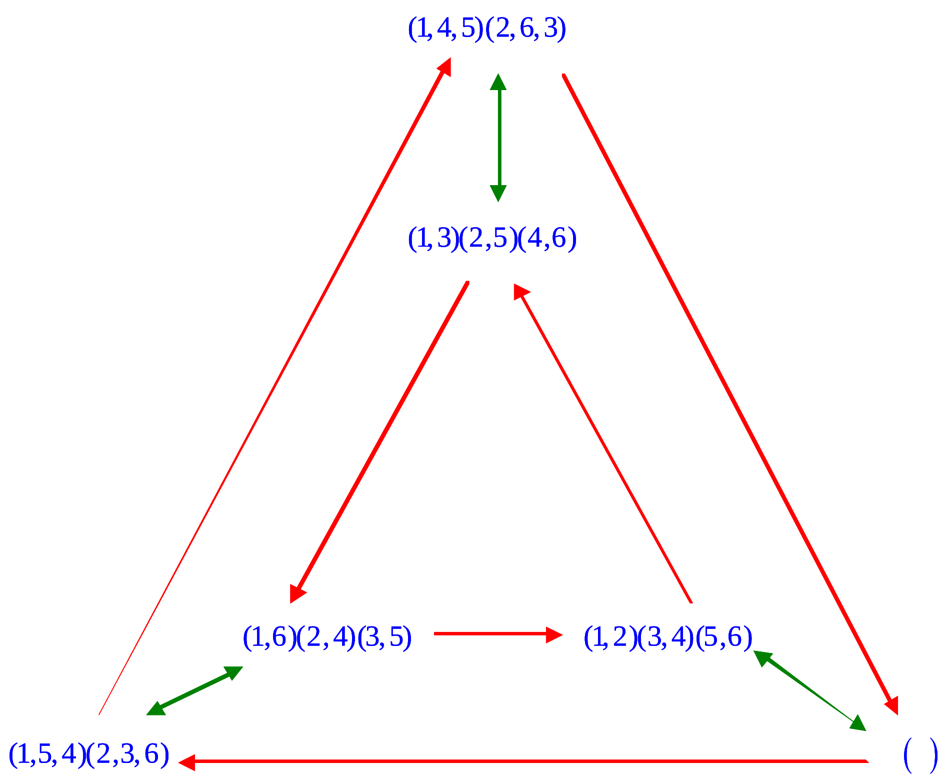
And since $(2,3)(1,2,3) = (1,2)$, let's also find the permutation that corresponds to the cycle $(1,2)$ by using the table below to obtain $(1,3)(2,5)(4,6)$.

Position

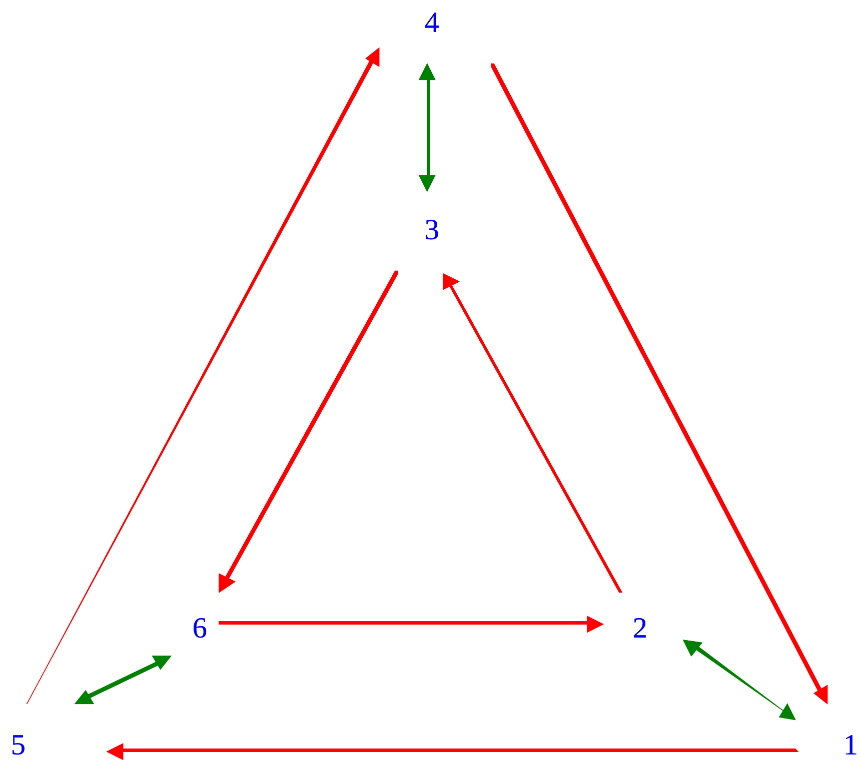
1 2 3 4 5 6

```
*        | ()      (2,3)   (1,2)   (1,2,3) (1,3,2) (1,3)
--------+------------------------------------------------
()       | ()      (2,3)   (1,2)   (1,2,3) (1,3,2) (1,3)
(2,3)    | (2,3)   ()      (1,2,3) (1,2)   (1,3)   (1,3,2)
(1,2)    | (1,2)   (1,3,2) ()      (1,3)   (2,3)   (1,2,3)
(1,2,3)  | (1,2,3) (1,3)   (2,3)   (1,3,2) ()      (1,2)
(1,3,2)  | (1,3,2) (1,2)   (1,3)   ()      (1,2,3) (2,3)
(1,3)    | (1,3)   (1,2,3) (1,3,2) (2,3)   (1,2)   ()
```

If we now do the multiplication $(1,2)(3,4)(5,6)*(1,5,4)(2,3,6)$, then we wind up with $(1,3)(2,5)(4,6)$. In other words, the multiplication $(2,3)*(1,2,3)=(1,2)$ in the first *group* corresponds exactly to the multiplication $(1,2)(3,4)(5,6)*(1,5,4)(2,3,6)=(1,3)(2,5)(4,6)$ in the second *group*, and this is exactly how we set up an *isomorphism* between our original *group* and our second *group*. What's extremely important, though, is that whereas our original *group* looked at permutations of 3 objects, our second *group* deals with permutations of 6 objects where 6 is the actual number of elements in each *group*. Furthermore, just as $(1,2,3)$ and $(2,3)$ generate the elements of our first *group*, so do $(1,5,4)(2,3,6)$ and $(1,2)(3,4)(5,6)$ generate the elements of our isomorphic *group*. If we now construct a *Cayley diagram* for these generators, then we obtain the following.



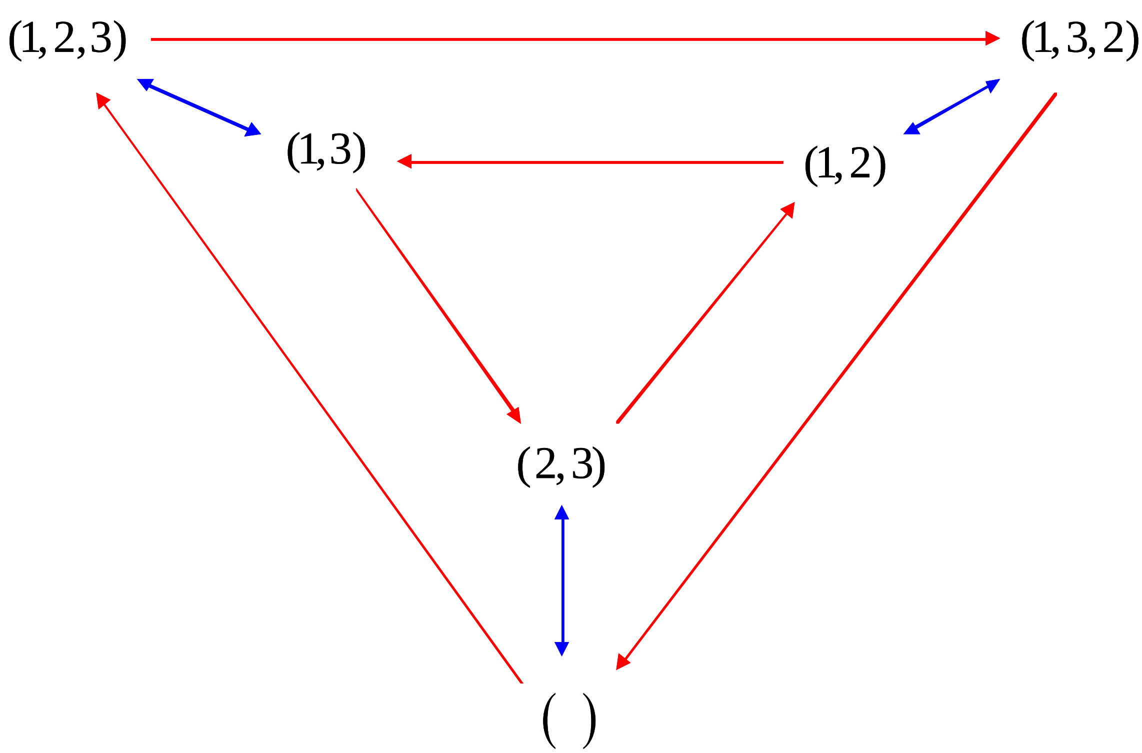
But on the other hand, if we replace our permutations by the appropriate position numbers, then we get the following *generator diagram*.



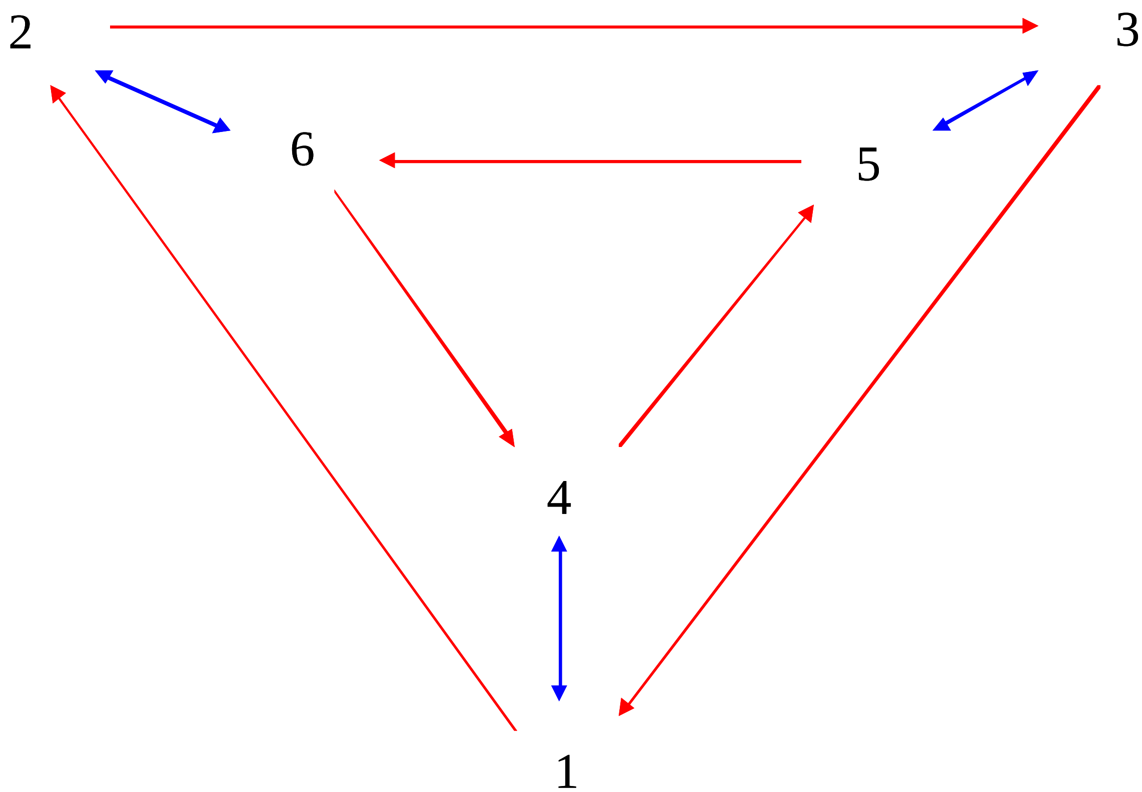
So what does all this mean? Let's go through the steps. We started with an equilateral triangle with vertices labeled 1, 2, & 3, and we constructed the six-element *dihedral group* for this geometric figure. Next, using the multiplication table for D_3, we converted this to an *isomorphic group* that acts on six elements that we labeled 1, 2, 3, 4, 5, & 6, and since our *group* D_3 has six elements, it is rather canonical to express it in terms of a *group* acting on six elements via permutations of those elements. Next we constructed both a *Cayley diagram* and a *generator diagram* for the *group* that acts on these six elements, and now the *Cayley diagram* and the *generator diagram* look essentially the same! And this shows us how, if we have a *group* of *n* elements, we can easily convert back and forth between a *generator diagram* of *n* elements and the corresponding permutations of those *n* elements. In this case, the *Cayley diagram* and the *generator diagram* are just two ways of looking at the same thing!

If we have a *group* of order *n* and if we represent it by either a *Cayley diagram* involving permutations of *n* elements or by a *generator diagram* involving permutations of *n* objects, then I'll refer to both as *canonical representations*. And now that we've seen how to develop such *canonical representations*, let me show you a shortcut that cuts through all the rigmarole. As before, we'll use a traditional *Cayley diagram* for D_3 as our starting point.

<u>Step 1:</u> Begin with any *Cayley diagram* for your *group*.



Step 2: Replace the elements by numbers in whatever way seems most convenient to you.

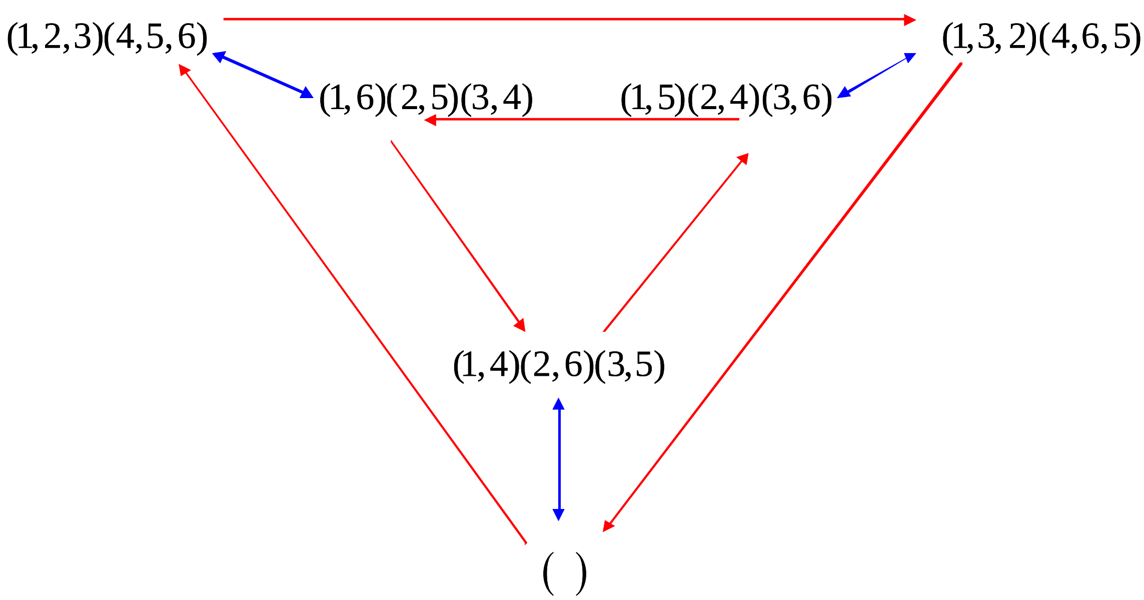


Step 3: Treat this is a *generator diagram*, and write down the permutations of the numbers that are indicated by the different colored paths.

$a = (1,2,3)(4,5,6)$

$b = (1,4)(2,6)(3,5)$

Step 5: Replace the numbers by permutations involving combinations of the permutations found in step 4 in order to create a *canonical Cayley diagram*.

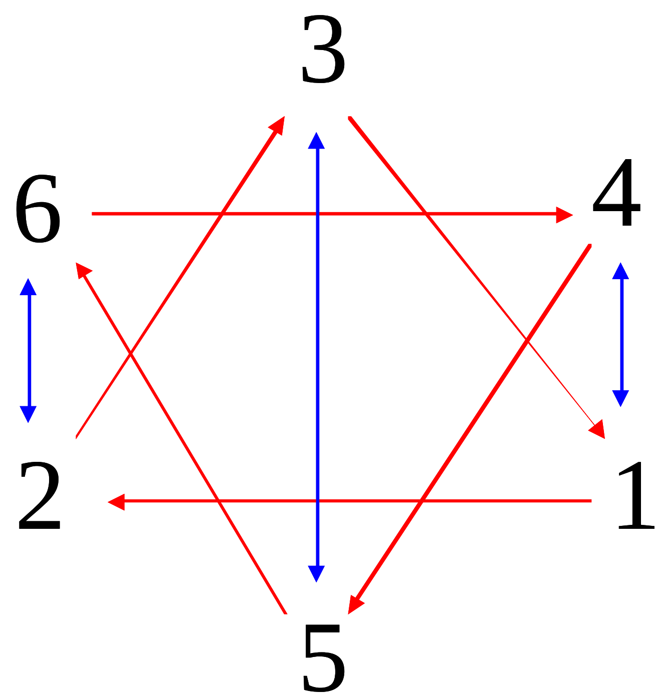


We can now use GAP software to verify that the *group* generated by our cycles *a* and *b* is indeed *isomorphic* to D_3 by generating both groups and examining the cycle structures of the elements in each group. Recall that for *groups* of order less than 16, each distinct *group* has a unique cycle structure. Thus, all we have to do, however, is verify that the elements of our new *group* have the same order or cycle length of the elements of our usual representation for D_3. We compare the two *groups* below with D_3 on the left and our new representation on the right.

$$\left\{\begin{matrix} (\ ) \\ (1,2) \\ (1,3) \\ (2,3) \\ (1,2,3) \\ (1,3,2) \end{matrix}\right\} \cong D_3 \qquad \left\{\begin{matrix} (\ ) \\ (1,4)(2,6)(3,5) \\ (1,6)(2,5)(3,4) \\ (1,5)(2,4)(3,6) \\ (1,2,3)(4,5,6) \\ (1,3,2)(4,6,5) \end{matrix}\right\}$$

From the above we can see that each *group* contains an element of order 1 (the identity), three elements of order 2, and two elements of order 3. Therefore, they are *isomorphic*!

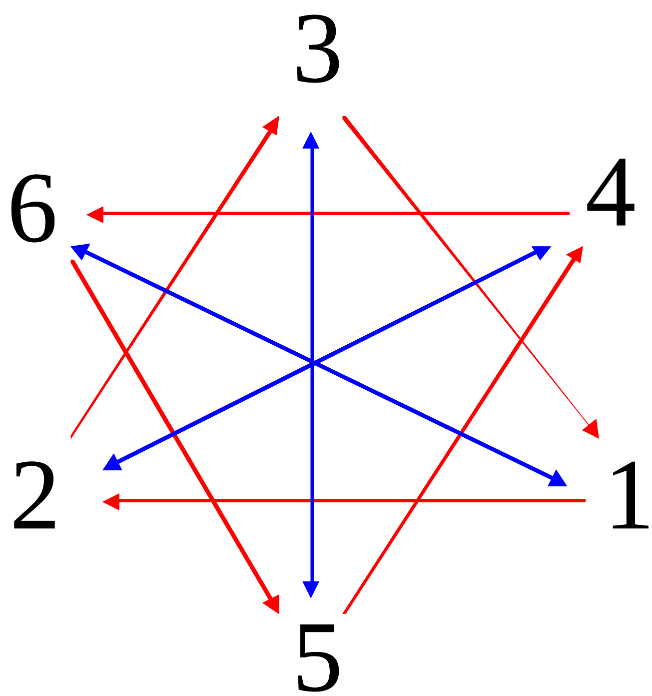
We can also rearrange the objects in our *generator diagram* to create results that may be more aesthetic to us. Hence, below are a couple of modified *generator diagrams* and generators that also generate *groups isomorphic* to D_3, as can be verified by examining the orders of the elements using GAP.



$$a = (1,2,3)(4,5,6)$$

$$b = (1,4)(2,6)(3,5)$$

$$\left\{\begin{array}{c} (\ ) \\ (1,4)(2,6)(3,5) \\ (1,6)(2,5)(3,4) \\ (1,5)(2,4)(3,6) \\ (1,2,3)(4,5,6) \\ (1,3,2)(4,6,5) \end{array}\right\}$$

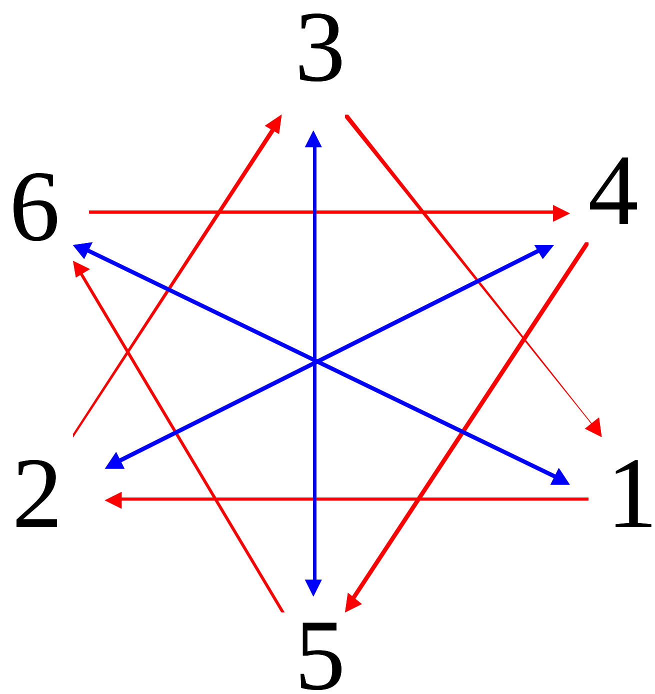


$$a = (1,2,3)(4,6,5)$$

$$b = (1,6)(2,4)(3,5)$$

$$\left\{\begin{matrix} (\ ) \\ (1,2,3)(4,6,5) \\ (1,3,2)(4,5,6) \\ (1,4)(2,5)(3,6) \\ (1,5)(2,6)(3,4) \\ (1,6)(2,4)(3,5) \end{matrix}\right\} \cong D_3$$

Notice, also, that if we reverse the direction of the arrows connecting, 4, 5, and 6, then the resulting *group* generated is now *isomorphic* to C_6, and while it may seem strange to have a *cyclic group* generated by two elements (in this case, one of order 2 and the other of order 3), it's all perfectly normal once we remember that $C_6 \cong C_2 \times C_3$.



$$a = (1,2,3)(4,5,6)$$

$$b = (1,6)(2,4)(3,5)$$

$$\left\{\begin{matrix} (\ ) \\ (1,2,3)(4,5,6) \\ (1,3,2)(4,6,5) \\ (1,4,3,6,2,5) \\ (1,5,2,6,2,4) \\ (1,6)(2,4)(3,5) \end{matrix}\right\} \cong C_6 \cong C_2 \times C_3$$

And now, we'll look at the same *groups* of small order that we examined in the previous installment (Part 4) of this work, but this time we'll present *cycle graphs*, *Cayley diagrams*, and *generator diagrams* for each *group*.

GROUPS OF ORDER 1

The only *group* of order 1 is the *group* that consists of a single element, the *identify element*. Consequently, it's a pretty simple *group*, and there is not much detail to give about it.

THE *IDENTITY GROUP*

Generators:

( )

Elements:

`{ () }`

Is *Abelian*?

Yes

Cycle Graph

( )

Cayley Diagram

$(\quad)$

Generator Diagram

1

GROUPS OF ORDER 2

Just as there is only one *group* of order 1, there is also only one *group*, up to *isomorphism*, of order 2. Also, when we use the phrase "*up to isomorphism*," recall that that means that even though we might use different names for the elements of the *group* and even though our *binary operations* may be defined differently in the different *groups*, the resulting multiplication tables all have the same algebraic structure. That means that we can take the elements of one *group*, translate them into elements of the other *group*, and then the corresponding elements will combine with one another in the same way. For example, below are four different looking multiplication tables that all represent the one (*up to isomorphism*) *group* of order 2.

	0	**1**
0	0	1
1	1	0

	1	**2**
1	1	2
2	2	1

	a	**b**
a	a	b
b	b	a

	no flip	**flip**
no flip	no flip	flip
flip	flip	no flip

For the last *group* multiplication table in our list, what we have in mind is a light switch and the 2-element *group* associated with it. Doing nothing, not flipping the switch at all, is the *identity element* in this *group*. The only other element in the *group* is represented by flipping the switch, and if we flip the switch twice, then the result is the same as not flipping the switch at all. In other words, "flip times flip = no flip."

THE *CYCLIC GROUP* OF ORDER 2

$$C_2 \cong \mathbb{Z}_2$$

Generators:

$(1,2)$

Elements:

$$\left\{ \begin{matrix} (\ ) \\ (1,2) \end{matrix} \right\} \cong C_2 \}$$

Is *Abelian*?

Yes

Cycle Graph

$$
\begin{tikzcd}
(1,2) \arrow[d, leftrightarrow] \\
(\ )
\end{tikzcd}
$$

Cayley Diagram

$$\begin{tikzcd} (1,2) \arrow[d, leftrightarrow] \\ (\ ) \end{tikzcd}$$

Generator Diagram

GROUPS OF ORDER 3

There is only one *group* of order 3, and it is the *cyclic group* C_3. Notice, too, that 3 is a prime number. Whenever the order of a *group* is a prime such as 2 or 3, then the only group of that order is going to be a *cyclic group*. This is because for *finite groups* the order of any *subgroup* has to be a divisor of the order of the *group*, and the only divisors of a prime number are itself and 1. Hence, the only *subgroups* of a *group* of prime order are the whole *group* and the *identity*, and they are also *normal subgroups*. Furthermore, C_3 is *simple* since it doesn't have any *normal subgroups* besides itself and the *identity*. Notice, also, that for any given finite order, there always exists a *cyclic group* of that order. Hence, when the order is prime, the only group that exists is the *cyclic group* of that prime order.

<u>THE *CYCLIC GROUP* OF ORDER 3</u>

$$C_3 \cong \mathbb{Z}_3$$

<u>Generators:</u>

$(1,2,3)$

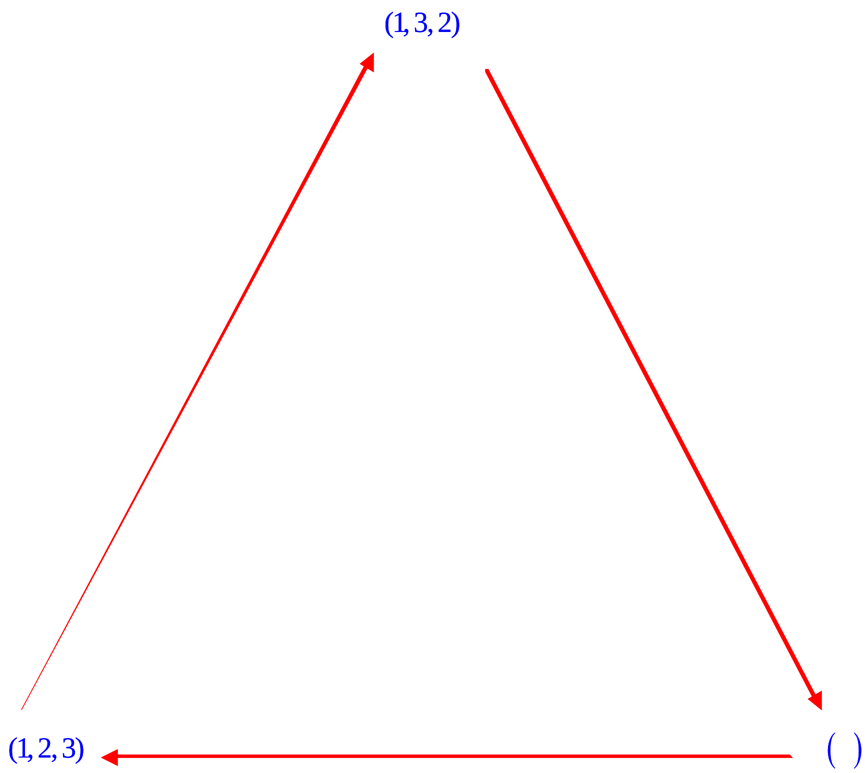
<u>Elements:</u>

$$\left\{\begin{matrix} (\ ) \\ (1,2,3) \\ (1,3,2) \end{matrix}\right\} \cong C_3$$

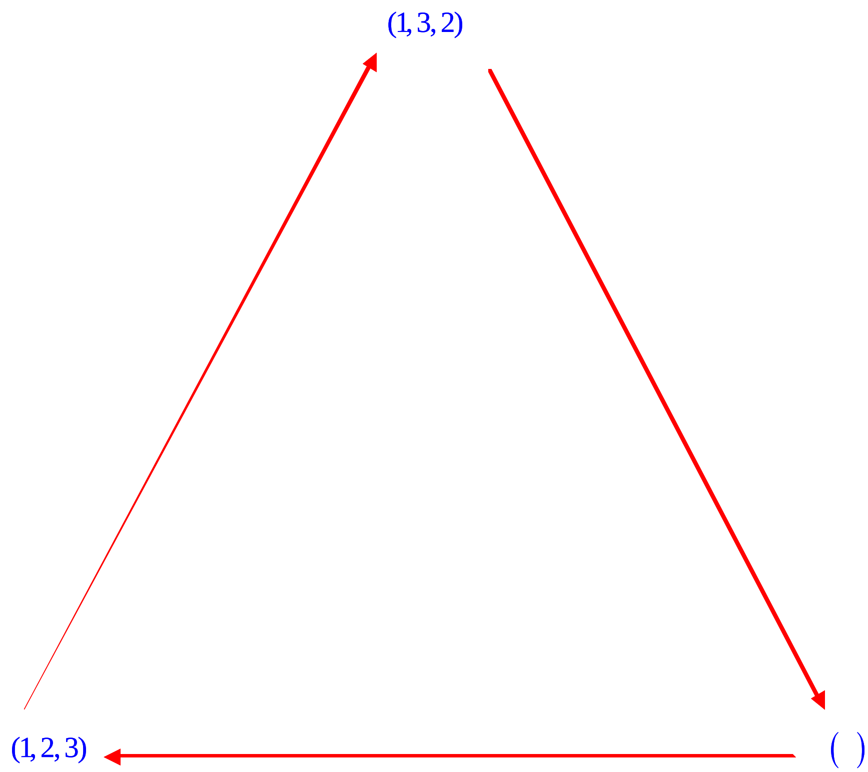
<u>Is *Abelian*?</u>

Yes

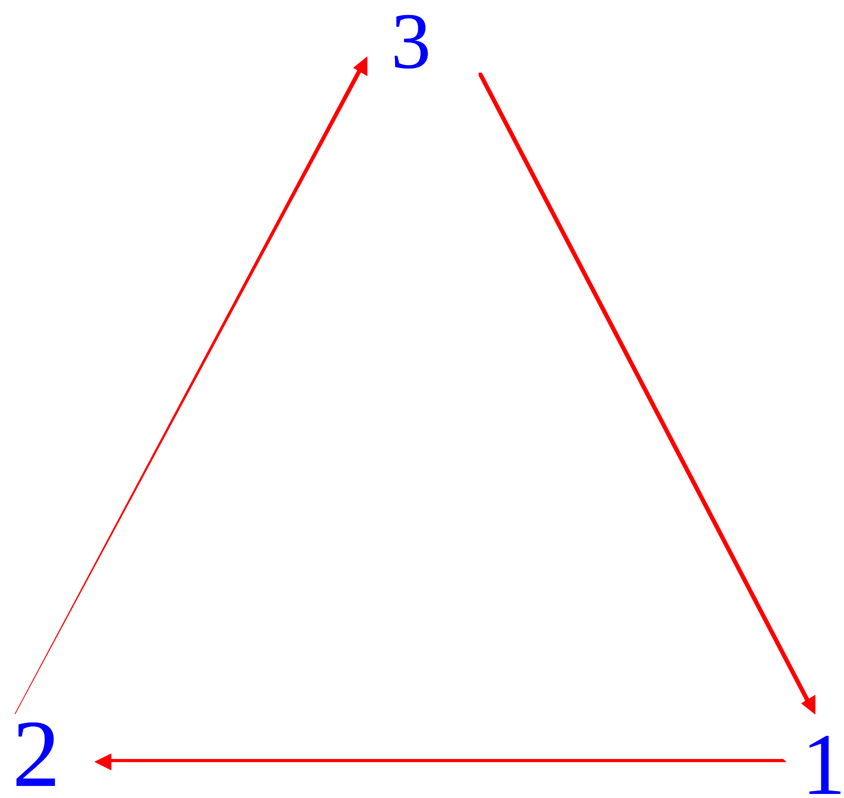
Cycle Graph



Cayley Diagram



Generator Diagram



GROUPS OF ORDER 4

There exist two *groups* of order 4 and both are *abelian*. Consequently, we can apply the *Fundamental Theorem of Finite Abelian Groups* which tells us that each *group* can be expressed as a *direct product* of *cyclic groups* of prime power order. In this case that means that the only two possible *groups* are the *cyclic group* C_4 and the *direct product* $C_2 \times C_2$. The group $C_2 \times C_2$ is also known as the *Klein 4-group* or as *Vierergruppe* (German for *4-group*). Additionally, it is sometimes denoted by K_4 or by V, and a good representation for this *group* consists of two light switches each of which can be flipped on or off. Let f_1 represent flipping the first switch, let f_2 represent flipping the second switch, and let 0 represent no flip at all. Then using this notation we can represent the elements of the group as $\{(0,0),(f_1,0),(0,f_2),(f_1,f_2)\}$ where $f_1^{\,2} = 0 = f_2^{\,2}$.

<u>THE *CYCLIC GROUP* OF ORDER 4</u>

$$C_4 \cong \mathbb{Z}_4$$

<u>Generators:</u>

$a = (1,2,3,4)$

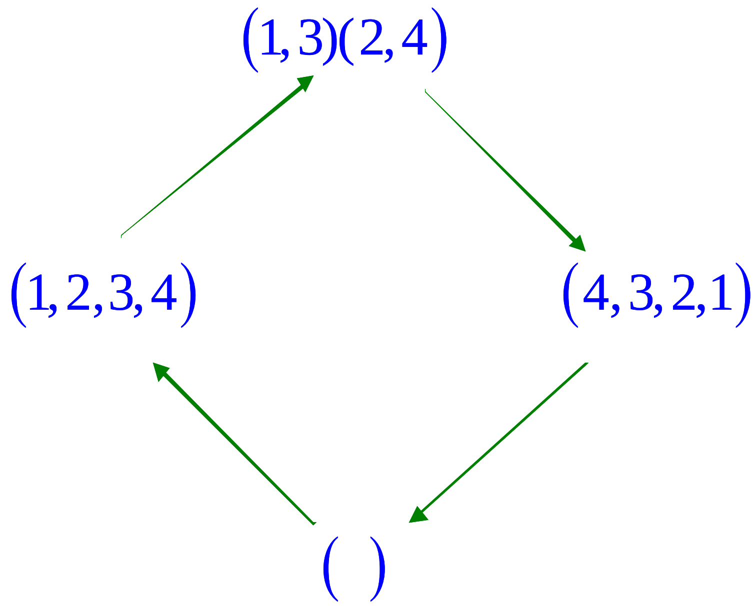
<u>Elements:</u>

$$\left\{\begin{matrix} (\ ) \\ (1,2,3,4) \\ (1,3)(2,4) \\ (4,3,2,1) \end{matrix}\right\} \cong C_4$$

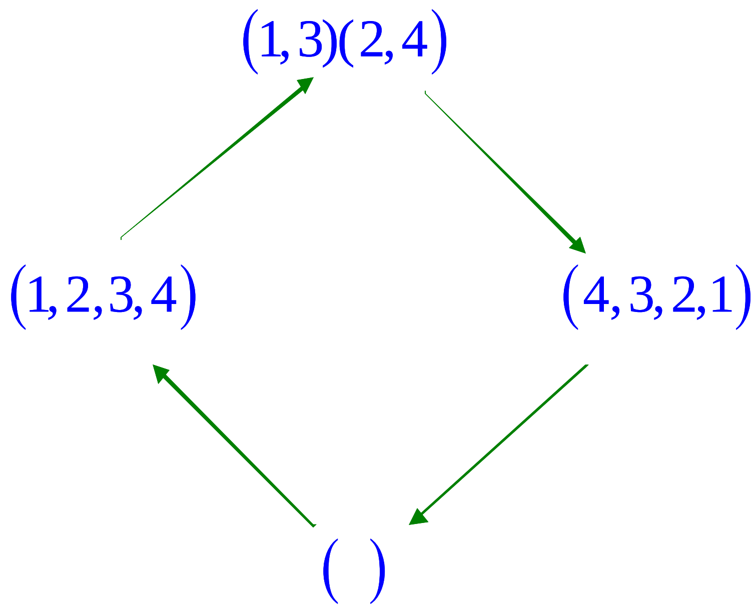
<u>Is *Abelian*?</u>

Yes

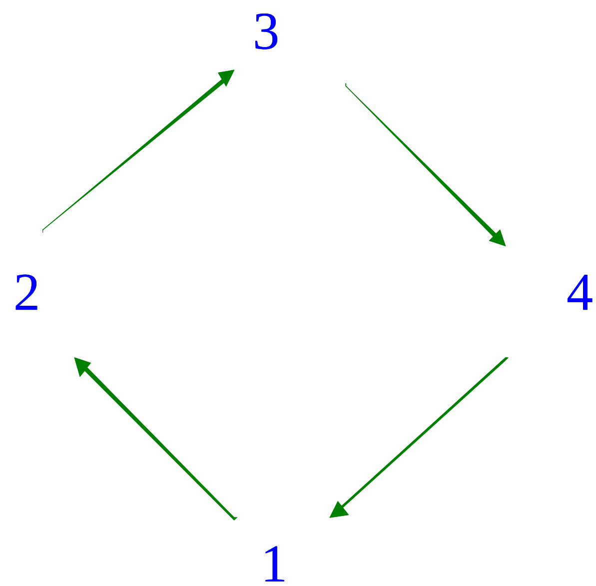
Cycle Graph



Cayley Diagram



Generator Diagram



$$a = (1, 2, 3, 4)$$

THE *KLEIN 4-GROUP*

$$C_2 \times C_2 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$$

Generators:

$a = (1,2)$
$b = (3,4)$

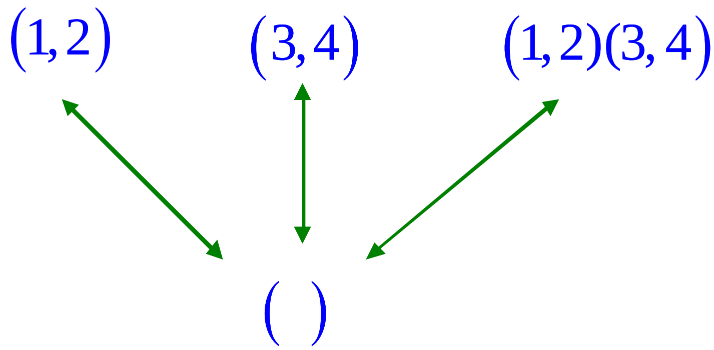
Elements:

$$\left\{ \begin{array}{c} (\ ) \\ (1,2) \\ (3,4) \\ (1,2)(3,4) \end{array} \right\} \cong C_2 \times C_2$$

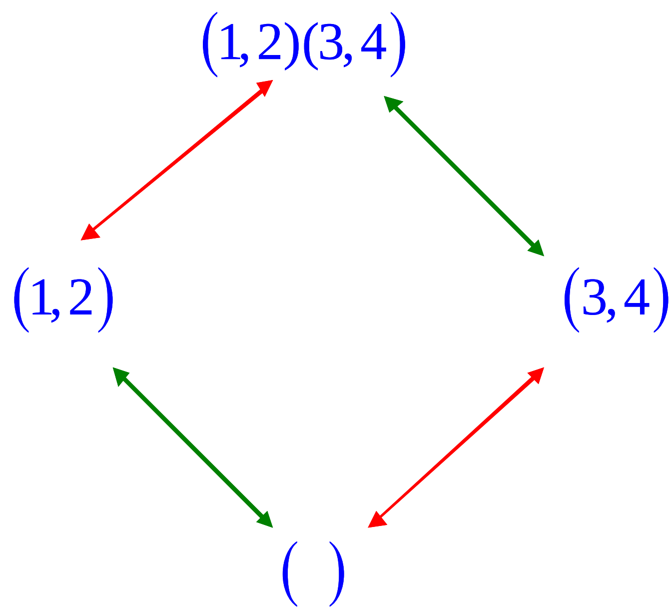
Is *Abelian*?

Yes

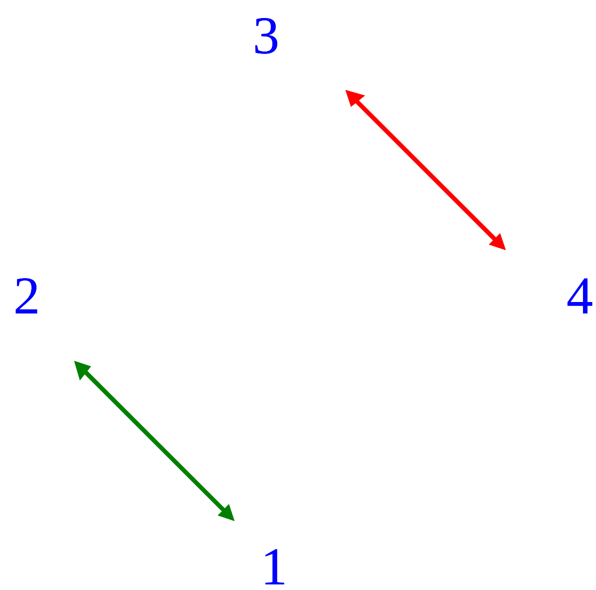
Cycle Graph



Cayley Diagram

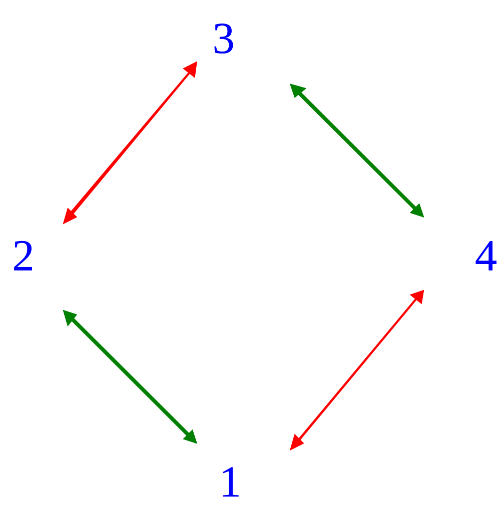


Generator Diagram



$$a = (1, 2)$$
$$b = (3, 4)$$

Or



$$a = (1, 2)(3, 4)$$
$$b = (1, 4)(2, 3)$$

GROUPS OF ORDER 5

Since 5 is a prime number, the only *group* that exists of order 5 is the *abelian cyclic group* of order 5, C_5. Furthermore, this *group* is simple since its only *normal subgroups* are itself and the *identity*.

THE *CYCLIC GROUP* OF ORDER 5

$$C_5 \cong \mathbb{Z}_5$$

Generators:

$(1,2,3,4,5)$

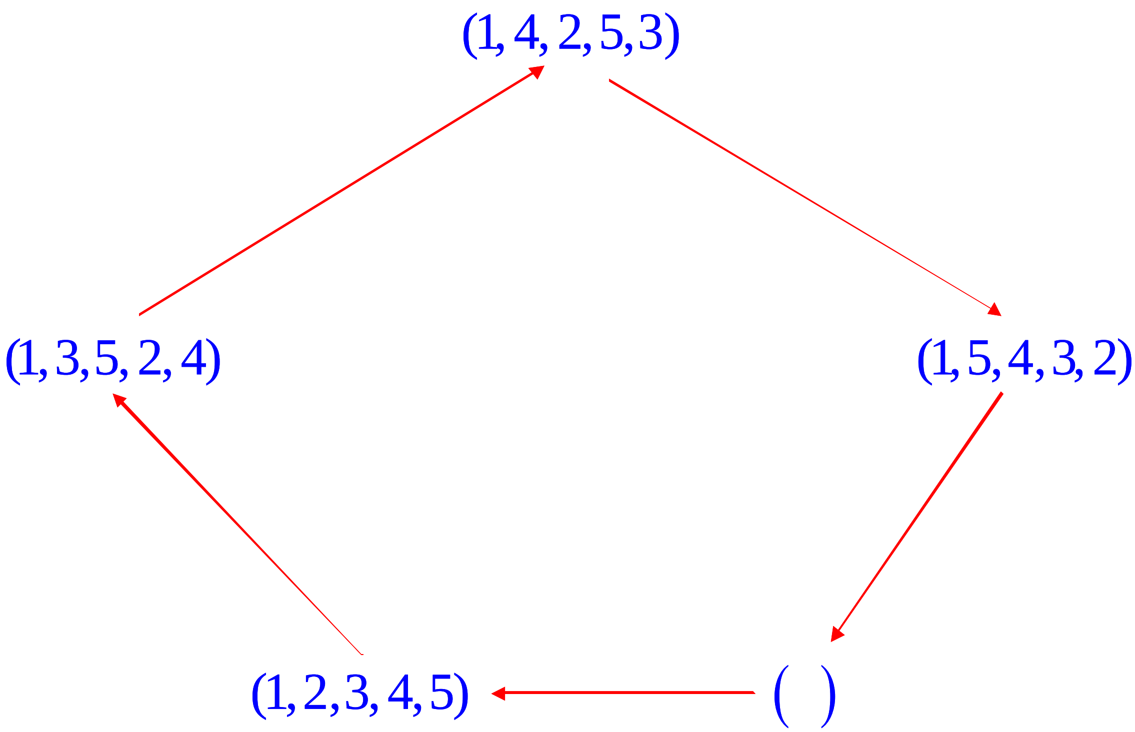
Elements:

$$\left\{\begin{matrix} (\ ) \\ (1,2,3,4,5) \\ (1,3,5,2,4) \\ (1,4,2,5,3) \\ (1,5,4,3,2) \end{matrix}\right\} \cong C_5$$

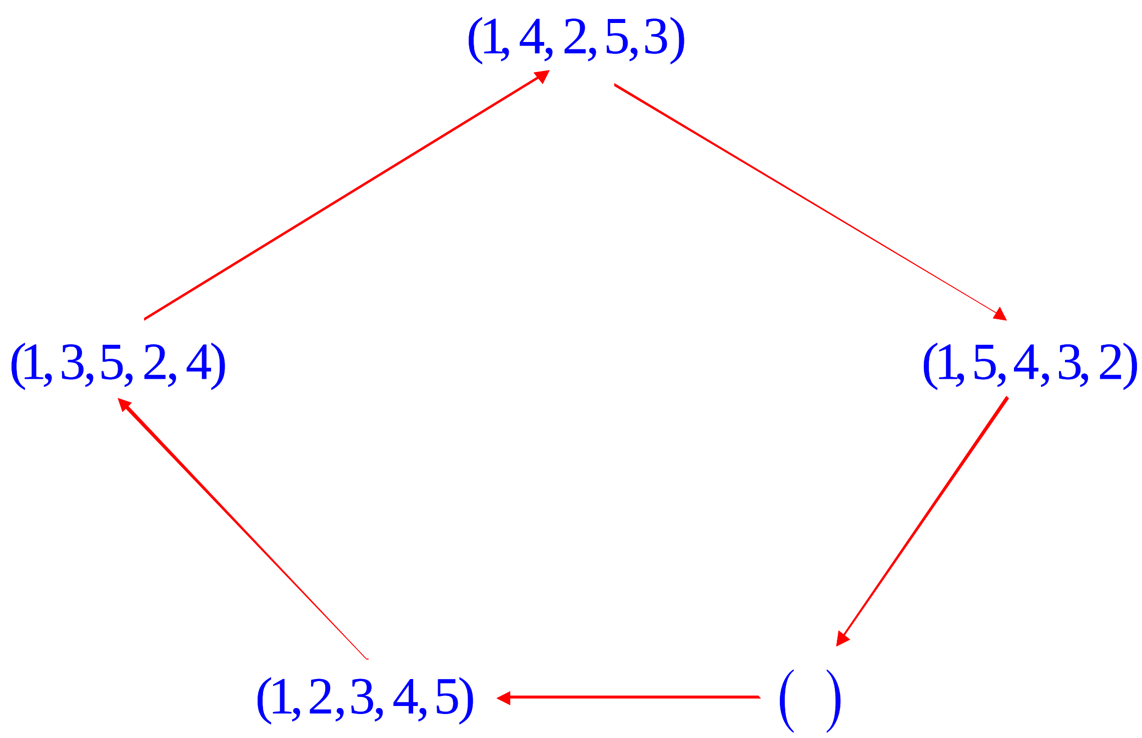
Is *Abelian*?

Yes

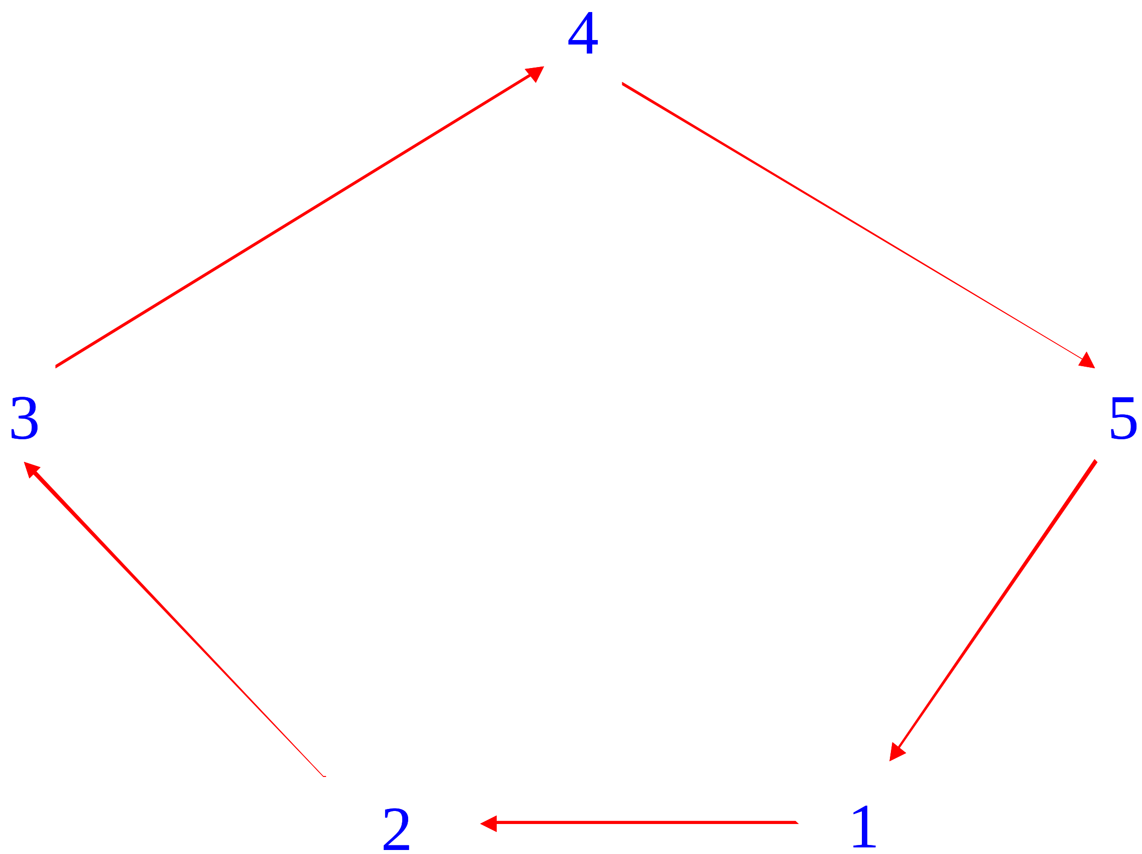
Cycle Graph



Cayley Diagram



Generator Diagram



Groups of Order 6

Order 6 for *groups* is very noticeable because this is the first time we encounter a *nonabelian group*! In fact, there exist just two *groups* of order 6 (two *groups* with six elements). One is the *cyclic group* of order 6, C_6, and the other is the *dihedral group* of degree 3, D_3. Notice that 6 is not a prime number, but that we can write 6 as 2×3 where 2 and 3 are relatively prime (that means that their only common factor is 1). When that happens with the order of a *cyclic group*, that means that we can also write our *cyclic group* as the *direct product* of smaller *cyclic groups*, and in this case we can write $C_6 \cong C_3 \times C_2$. The *dihedral group* D_3 has order 6, and recall that it represents the symmetries of an equilateral triangle. In other words, it is the *group* generated by rotations of our triangle through angles that are integer multiples of $120°$ and by flips about any of its three axes of symmetry. Furthermore, the number of permutations that can be made of 3 objects is 6, and that means that the symmetric group of degree 3, S_3, which is the *group* of all permutations that can be made of 3 objects is essentially identical or *isomorphic* with the *dihedral group* D_3, $D_3 \cong S_3$. Additionally, this is the only time something like this happens. Since the order of D_n is $2n$ and since the order of S_n is $n! = n(n-1)(n-2)\ldots(1)$, the only time these two computations are the same is when $n = 3$. Something else worth noting is that for any value of *n* there always exists a *cyclic group* of degree *n*, and for any value $2n$ where $n \geq 3$, there is always a *dihedral group*, D_n, of that order, and for any *dihedral group* D_n it is also true that $D_n \cong C_n >\!\!\lhd C_2$. Thus, $D_3 \cong S_3 \cong C_3 >\!\!\lhd C_2$. A lot of *groups* of higher order turn out to be either *cyclic* or *dihedral*. And if we add to this list the *symmetric groups*, *alternating groups*, *direct products*, and *semidirect products*, then those are probably the majority of the *groups* we are likely to encounter. Things will change though when we get to order 8 and discover an interesting *group* called the *Quaternion group* which is *nonabelian* and which falls into none of the aforementioned categories.

THE *CYCLIC GROUP* OF ORDER 6

$$C_6 \cong C_2 \times C_3 \cong \mathbb{Z}_2 \times \mathbb{Z}_3 \cong \mathbb{Z}_6$$

Generators:

$a = (1,2)$
$b = (3,4,5)$

or

$a = (1,2,3,4,5,6)$

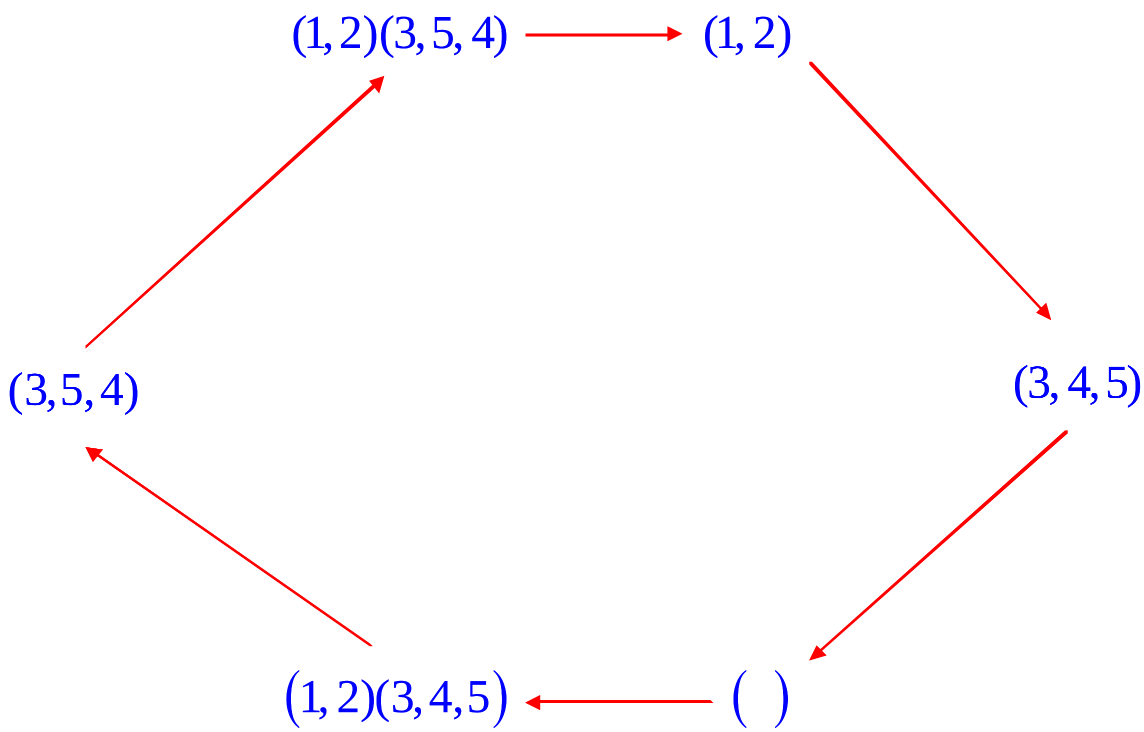
Elements:

$$\left\{ \begin{array}{c} (\ ) \\ (3,4,5) \\ (3,5,4) \\ (1,2) \\ (1,2)(3,4,5) \\ (1,2)(3,5,4) \end{array} \right\} \cong \left\{ \begin{array}{c} (\ ) \\ (1,2,3,4,5,6) \\ (1,3,5)(2,4,6) \\ (1,4)(2,5)(3,6) \\ (1,5,3)(2,6,4) \\ (1,6,5,4,3,2) \end{array} \right\} \cong C_6$$

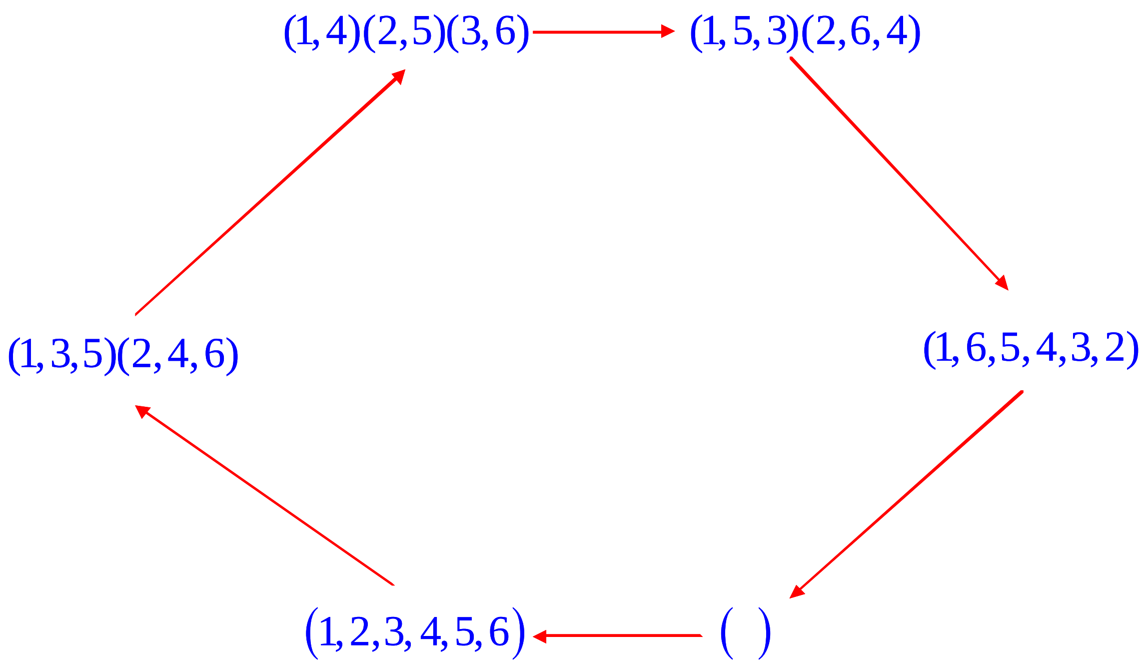
Is *Abelian*?

Yes

Cycle Graph

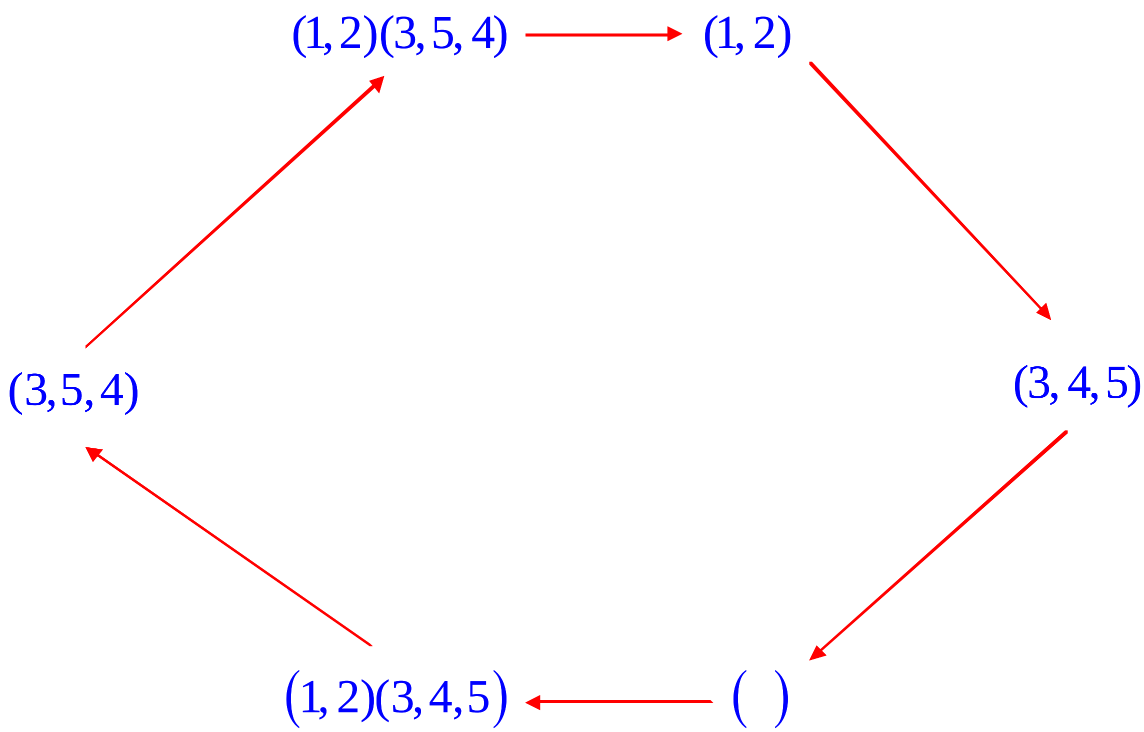


Or

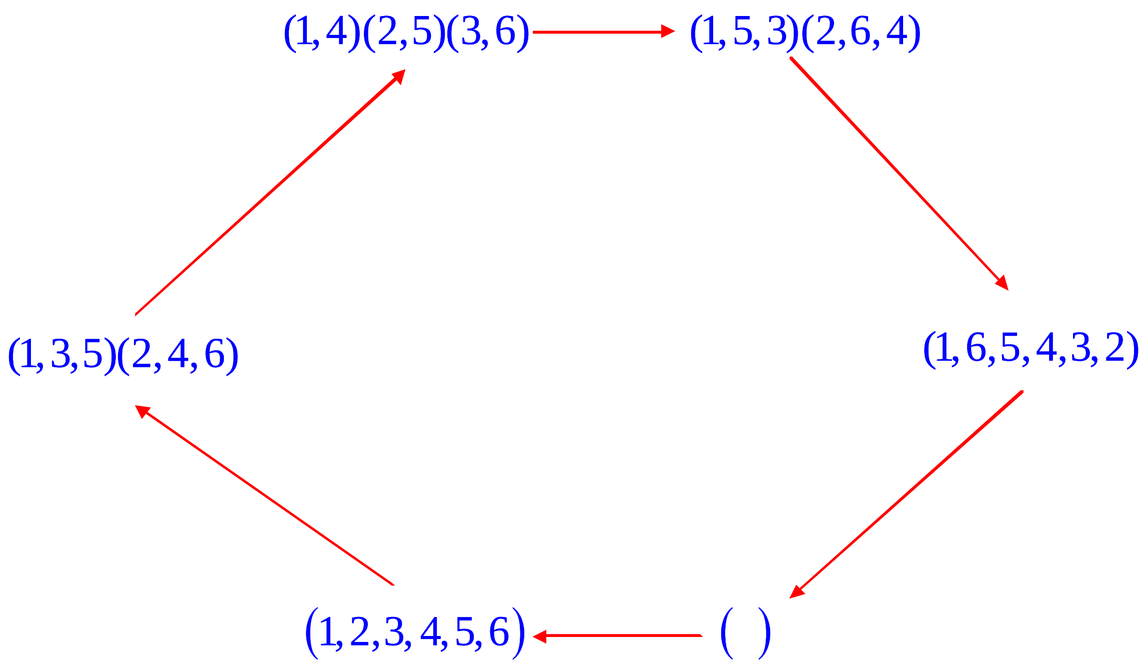


$(1,4)(2,5)(3,6)$
$(1,5,3)(2,6,4)$
$(1,3,5)(2,4,6)$
$(1,6,5,4,3,2)$
$(1,2,3,4,5,6)$
$(\ )$

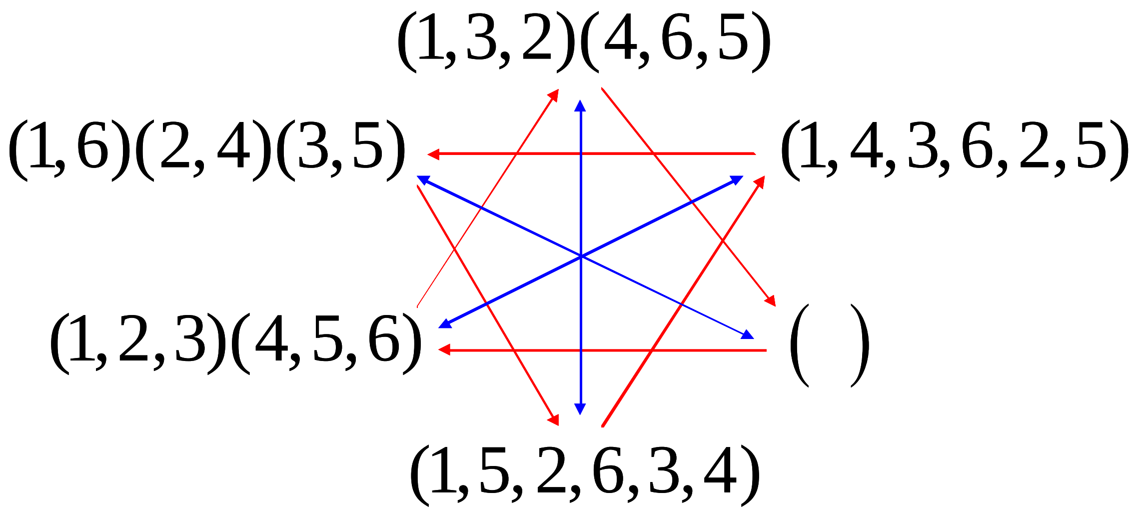
Cayley Diagram



or

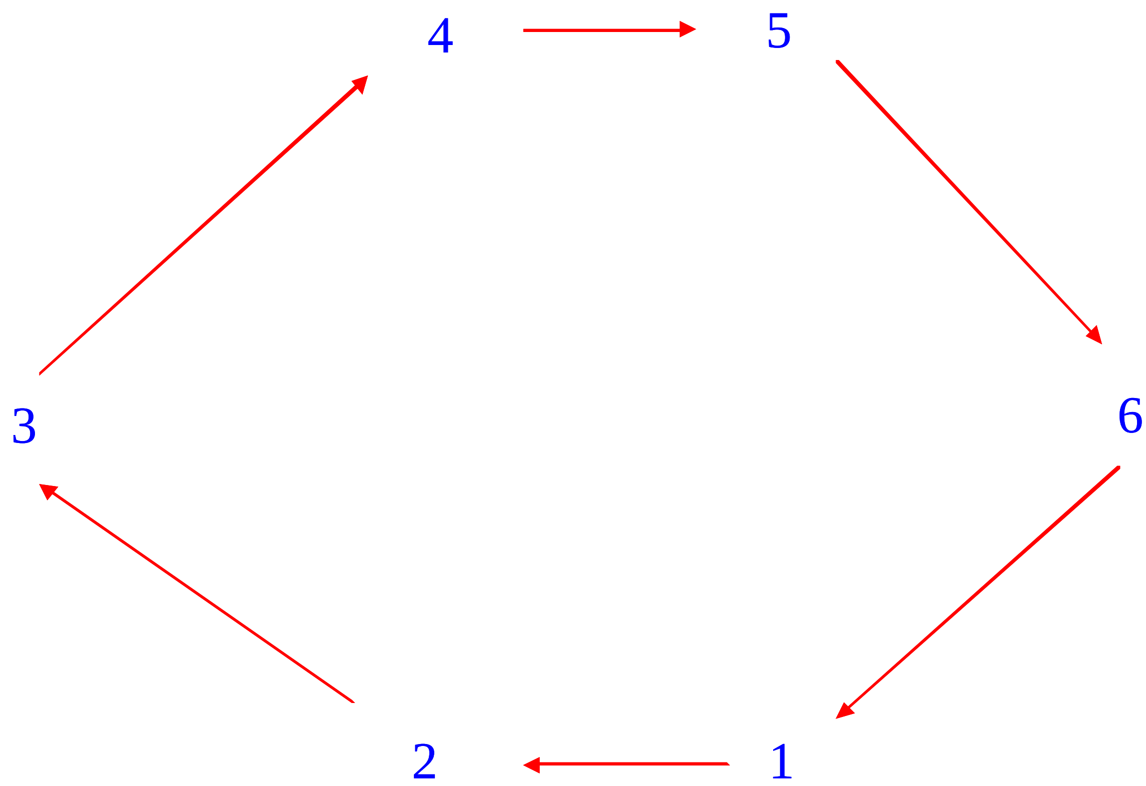


or

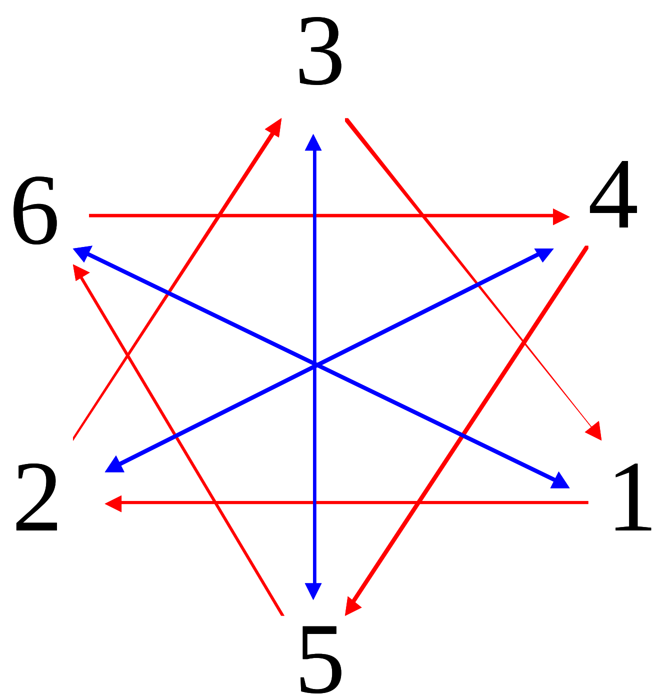


(1,3,2)(4,6,5)
(1,6)(2,4)(3,5)
(1,4,3,6,2,5)
(1,2,3)(4,5,6)
( )
(1,5,2,6,3,4)

Generator Diagram



or


3
6
4
2
1
5

THE *DIHEDRAL/SYMMETRIC GROUP* OF ORDER 6

$$D_3 \cong S_3 \cong \mathbb{Z}_3 > \triangleleft\, \mathbb{Z}_2 \cong C_3 > \triangleleft\, C_2$$

Generators:

$a = (1,2,3)$
$b = (2,3)$

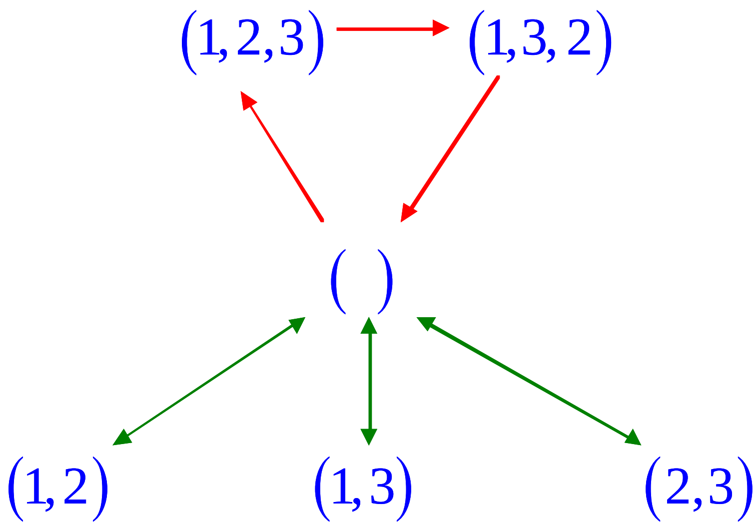
Elements:

$$\left\{ \begin{array}{c} (\ ) \\ (1,2) \\ (1,3) \\ (2,3) \\ (1,2,3) \\ (1,3,2) \end{array} \right\} \cong D_3$$

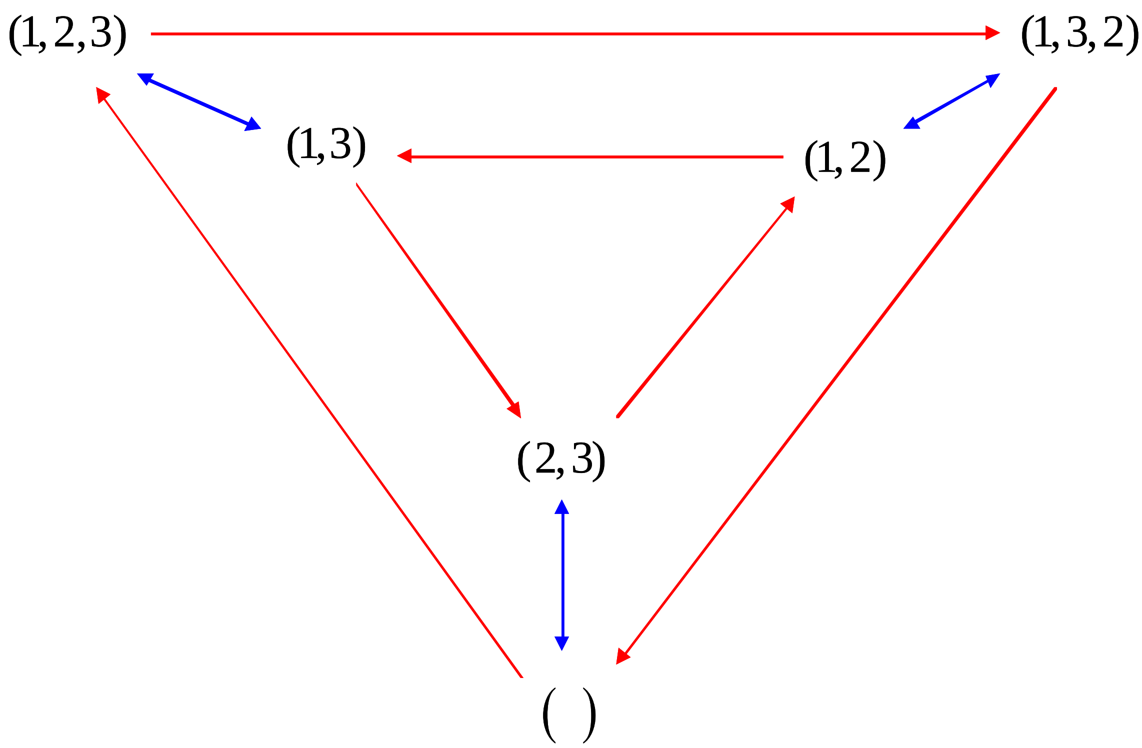
Is *Abelian*?

No

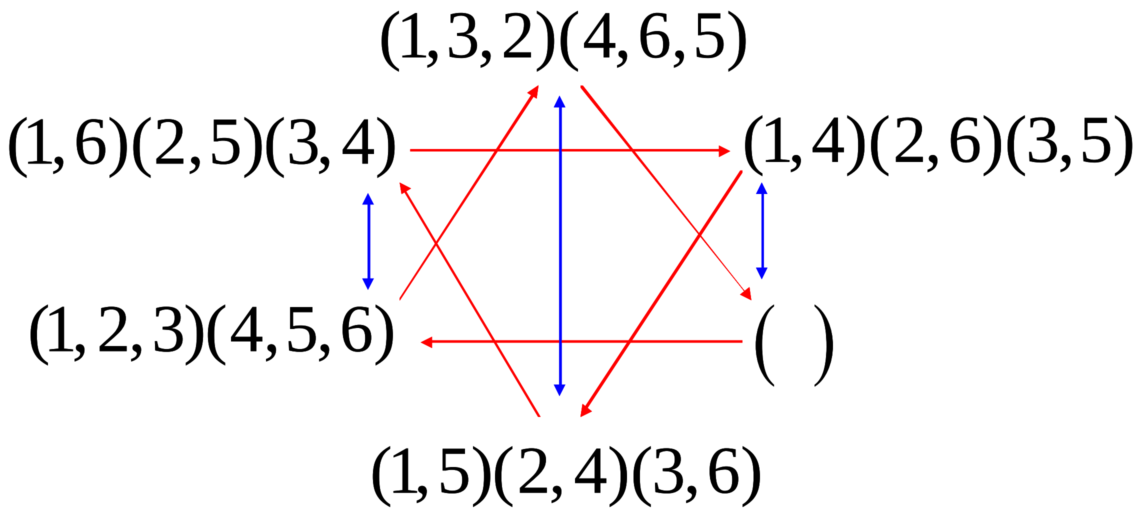
Cycle Graph



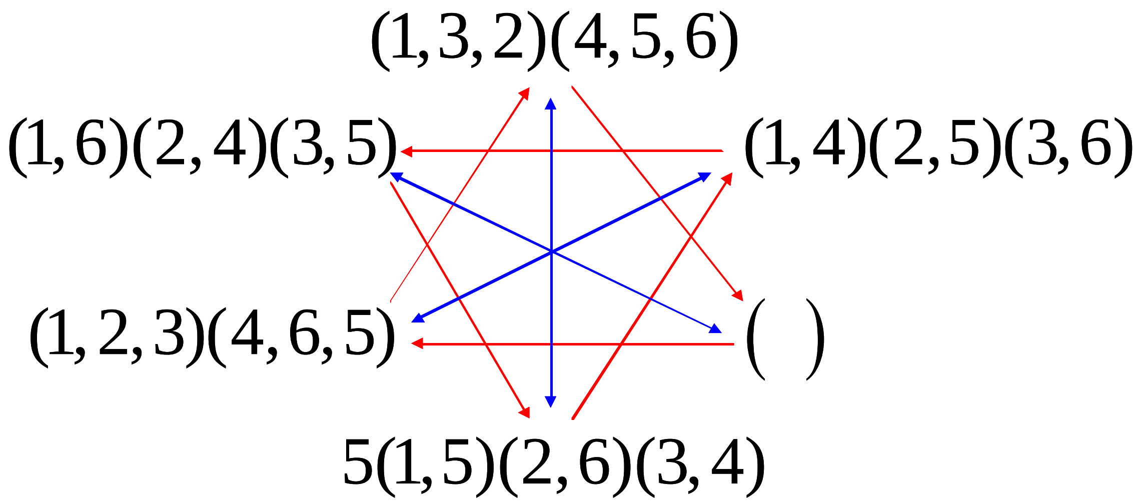
Cayley Graph



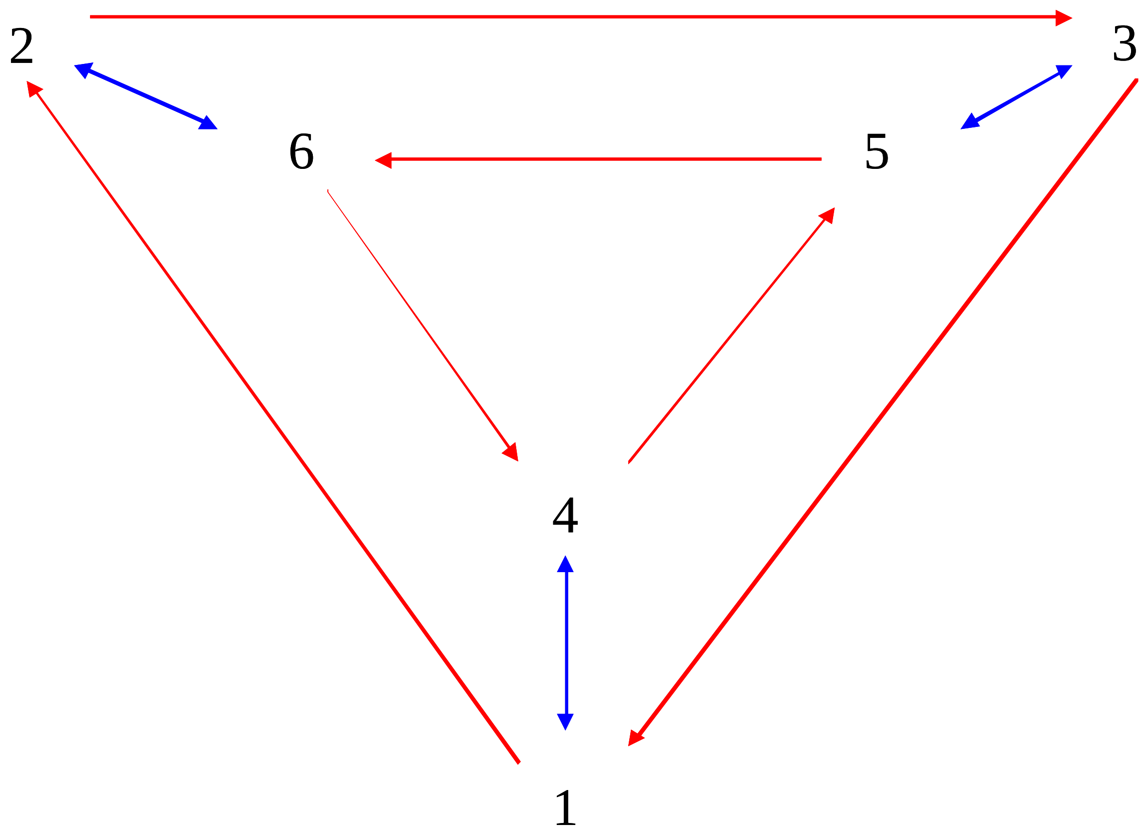
or



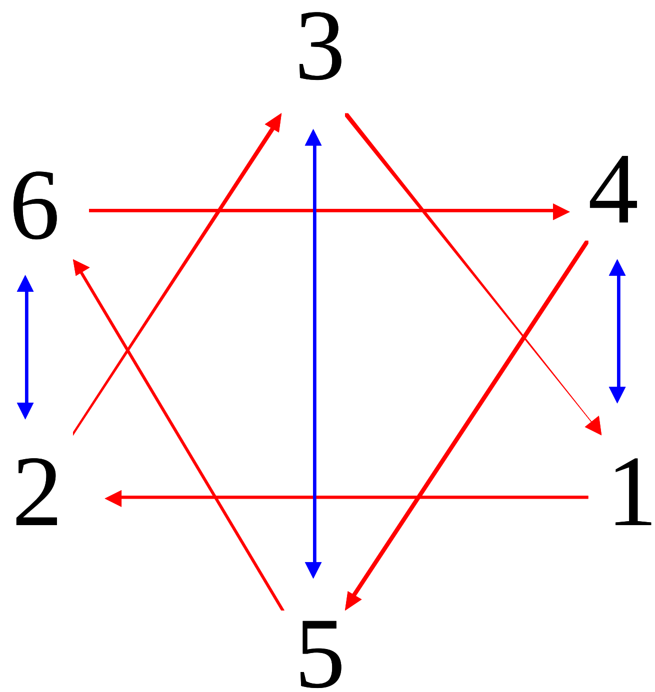
Or


(1,3,2)(4,5,6)
(1,6)(2,4)(3,5)
(1,4)(2,5)(3,6)
(1,2,3)(4,6,5)
( )
5(1,5)(2,6)(3,4)

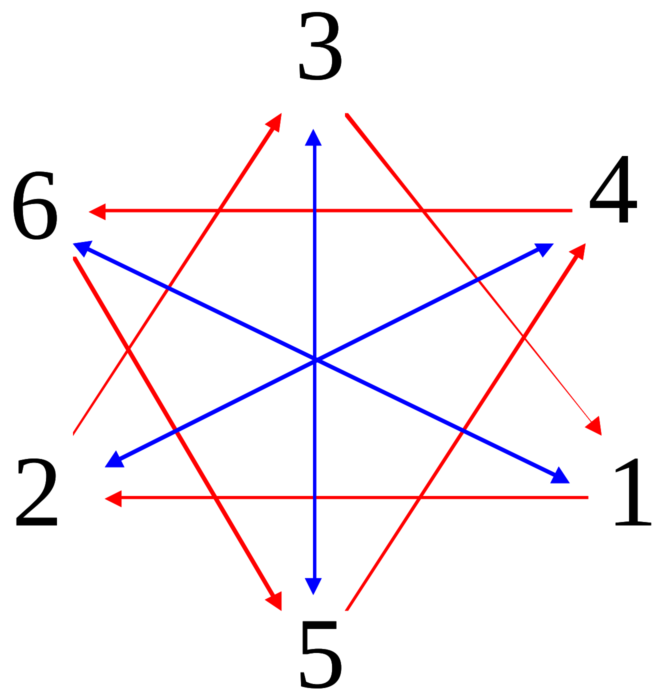
Generator Diagram



or



or


3
6
4
2
1
5

GROUPS OF ORDER 7

The number 7 is prime, so you know what that means. There exists only one *group* of order 7, and that is C_7, the *cyclic group* of order 7. Furthermore, again since 7 is prime, its only *subgroups* are itself and the *identity*.

THE *CYCLIC GROUP* OF ORDER 7

$$C_7 \cong \mathbb{Z}_7$$

Generators:

$(1,2,3,4,5,6,7)$

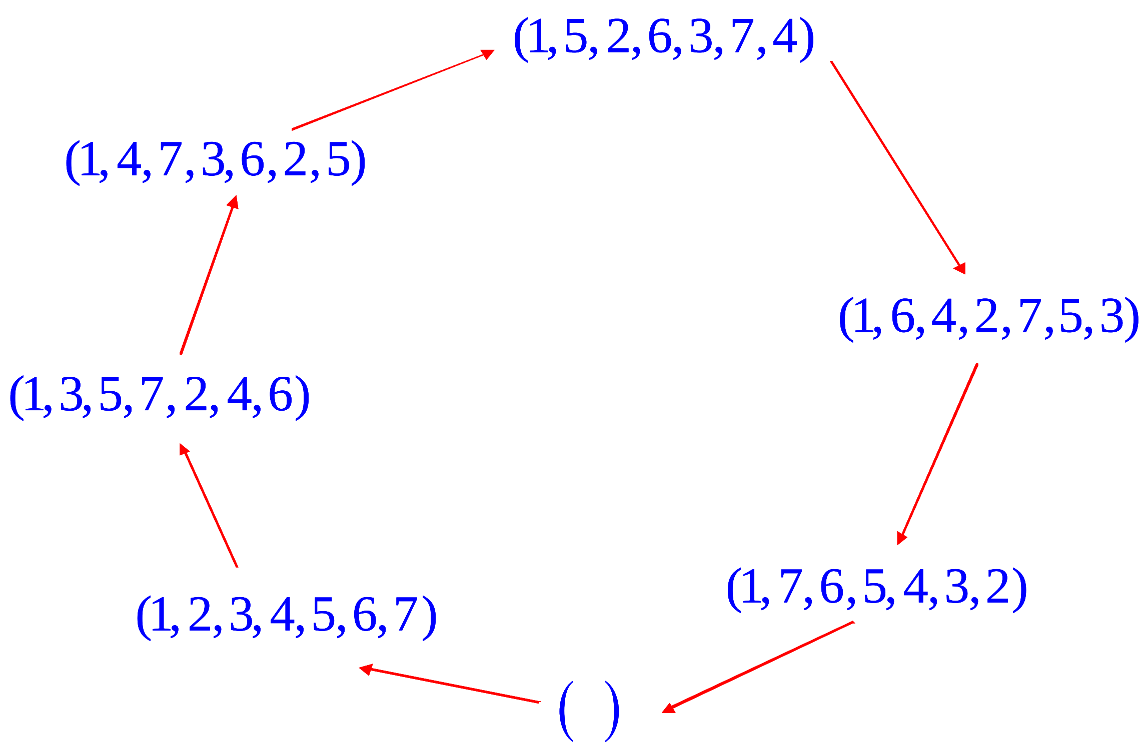
Elements:

$$\left\{\begin{matrix} (\ ) \\ (1,2,3,4,5,6,7) \\ (1,3,5,7,2,4,6) \\ (1,4,7,3,6,2,5) \\ (1,5,2,6,3,7,4) \\ (1,6,4,2,7,5,3) \\ (1,7,6,5,4,3,2) \end{matrix}\right\} \cong C_7$$

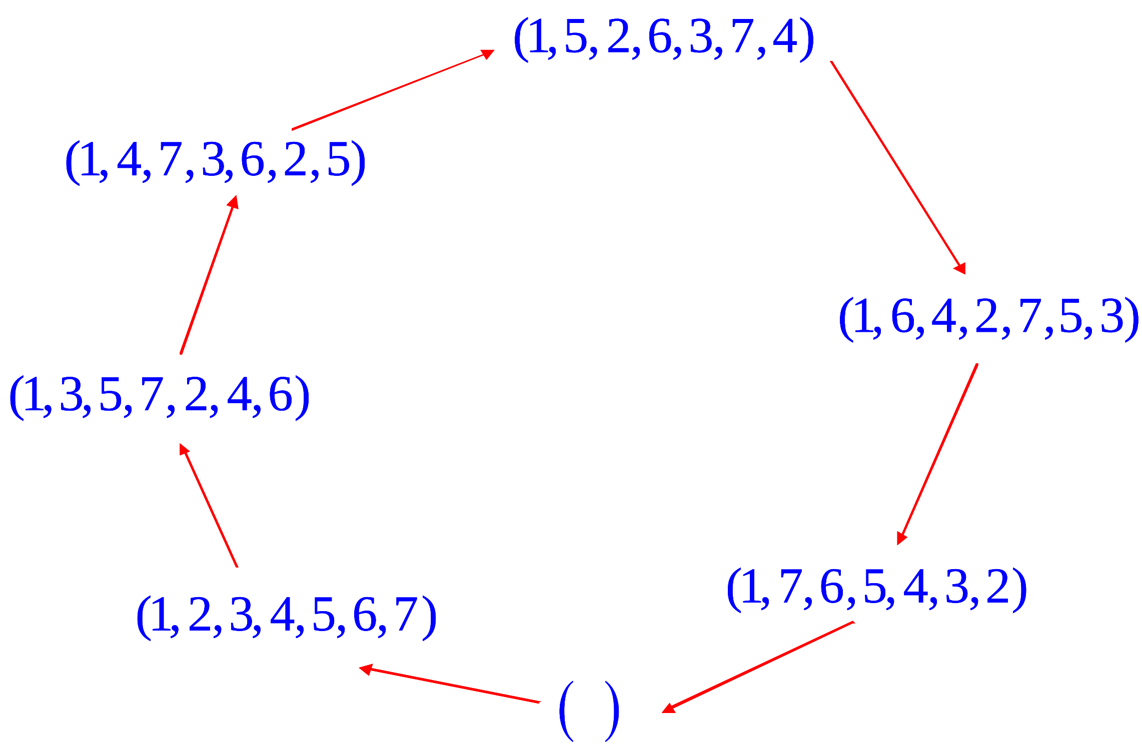
Is *Abelian*?

Yes

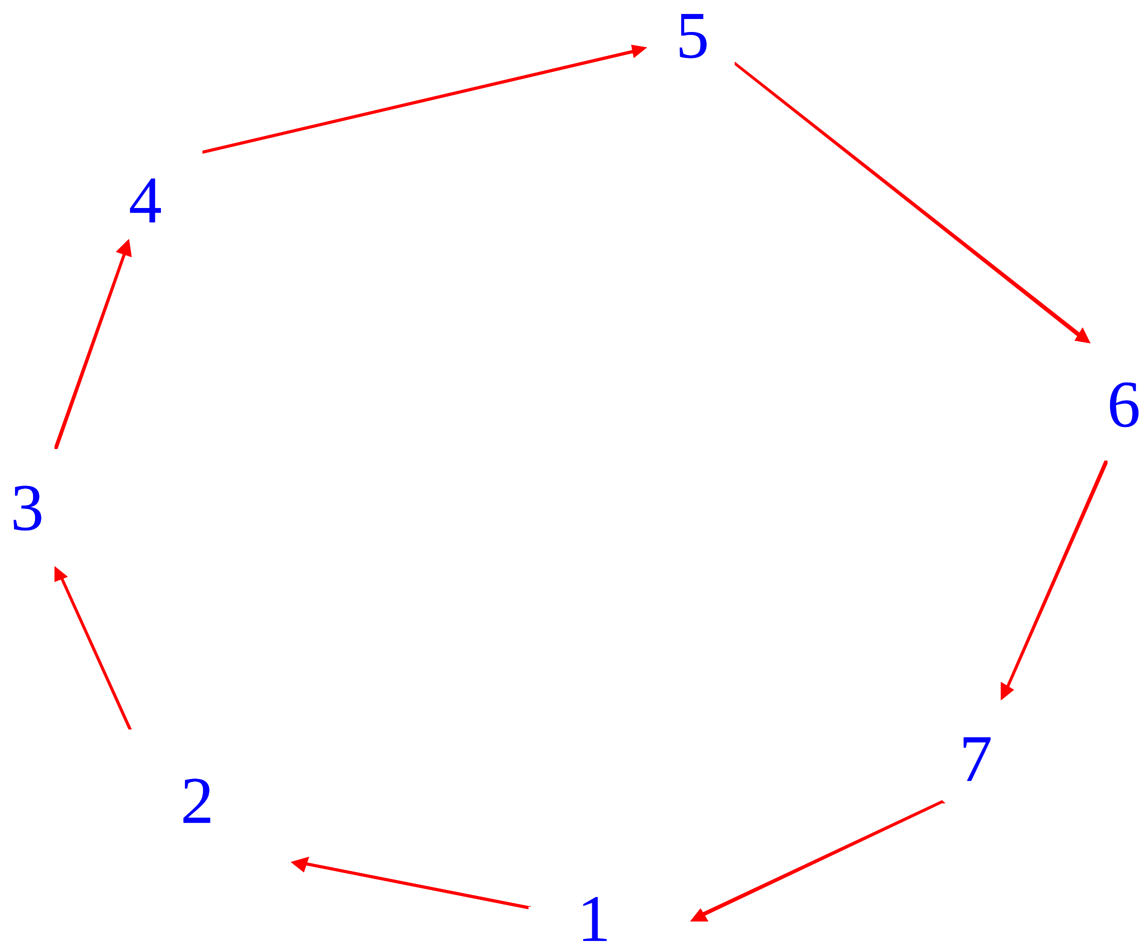
Cycle Graph



Cayley Diagram



Generator Diagram



GROUPS OF ORDER 8

Things get quite interesting once we get to 8. There exist five *groups* of order 8, and three of them are *abelian*. And by the *Fundamental Theorem of Finite Abelian Groups*, we can immediately identify the *abelian groups* as C_8, $C_4 \times C_2$, and $C_2 \times C_2 \times C_2$. Of the two *nonabelian groups*, since 8 is even we automatically know that one of them is D_4. The other *nonabelian group*, though, is called the *Quaternion group*, and it is quite interesting since it is not one of our usual *cyclic*, *dihedral*, *symmetric*, *alternating*, *direct product*, or *semidirect product groups*. It is something quite different, and notable feature of this *group* is that all of its *subgroups* are *normal* in spite of it being *nonabelian*. Also of interest is that *quaternions* were invented by the mathematician William Rowan Hamilton (1805-1865) as an extension of both *vectors* and *imaginary numbers*. Thus, we have *i, j,* and *k* which resemble the *unit vectors* studied in trigonometry and advanced calculus, and these quantities are also like *imaginary numbers* since $i^2 = j^2 = k^2 = -1$. When I was younger, *quaternions* weren't studied that much anymore, but these days there is renewed interest in the topic since they have turned out to be a useful mathematical tool for creating the kinds of computer generated effects that appear in many of today's movies.

THE *CYCLIC GROUP* OF ORDER 8

$$C_8 \cong \mathbb{Z}_8$$

Generators:

$(1,2,3,4,5,6,7,8)$

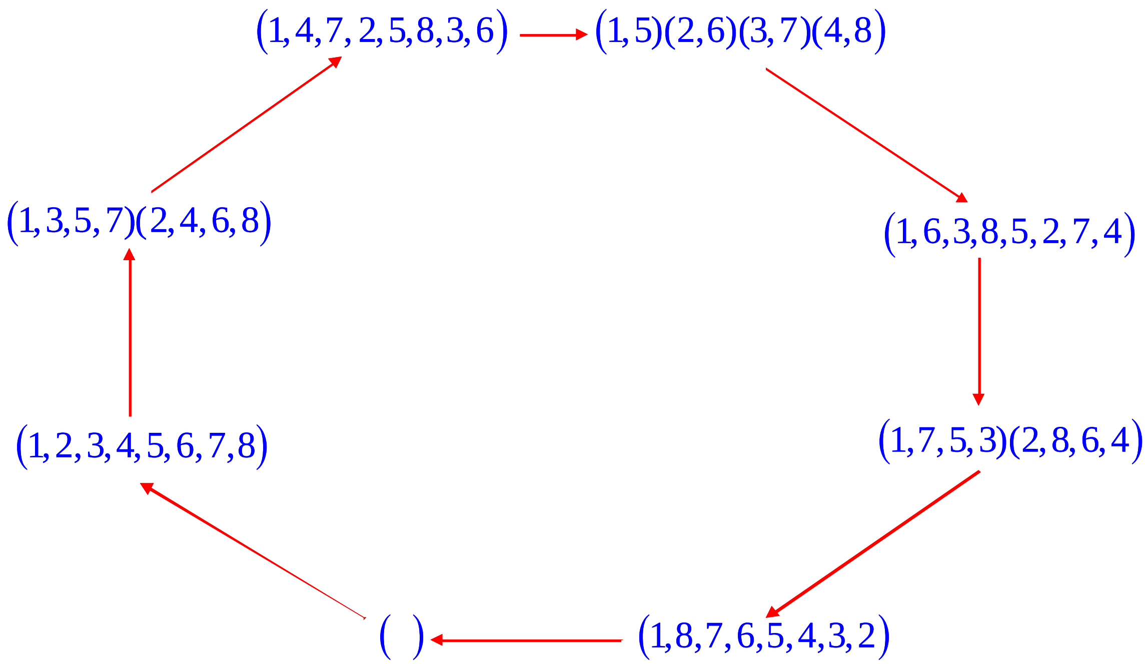
Elements:

$$\left\{\begin{array}{c} (\ ) \\ (1,2,3,4,5,6,7,8) \\ (1,3,5,7)(2,4,6,8) \\ (1,4,7,2,5,8,3,6) \\ (1,5)(2,6)(3,7)(4,8) \\ (1,6,3,8,5,2,7,4) \\ (1,7,5,3)(2,8,6,4) \\ (1,8,7,6,5,4,3,2) \end{array}\right\} \cong C_8$$

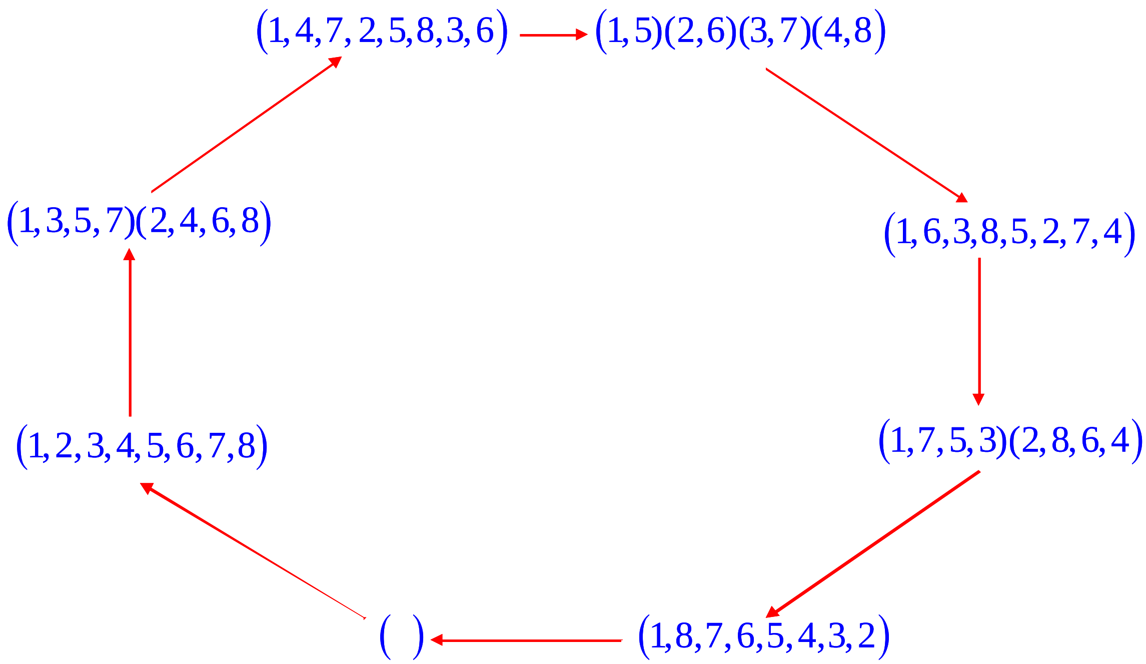
Is *Abelian*?

Yes

Cycle Graph

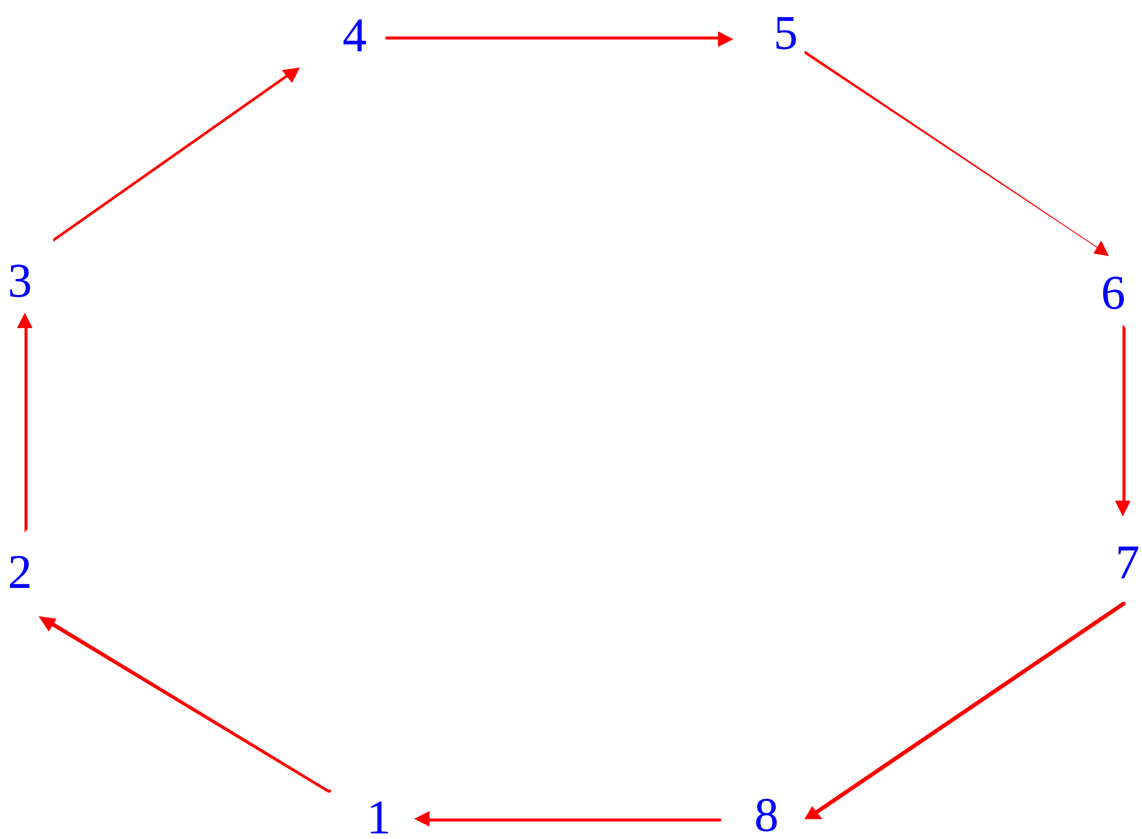

(1,4,7,2,5,8,3,6)
(1,5)(2,6)(3,7)(4,8)
(1,3,5,7)(2,4,6,8)
(1,6,3,8,5,2,7,4)
(1,2,3,4,5,6,7,8)
(1,7,5,3)(2,8,6,4)
( )
(1,8,7,6,5,4,3,2)

Cayley Diagram



$a = (1,2,3,4,5,6,7,8)$

Generator Diagram



$a = (1, 2, 3, 4, 5, 6, 7, 8)$

THE *DIRECT PRODUCT* $C_2 \times C_4$

$$C_2 \times C_4 \cong \mathbb{Z}_2 \times \mathbb{Z}_4$$

Generators:

$(1,2),(3,4,5,6)$

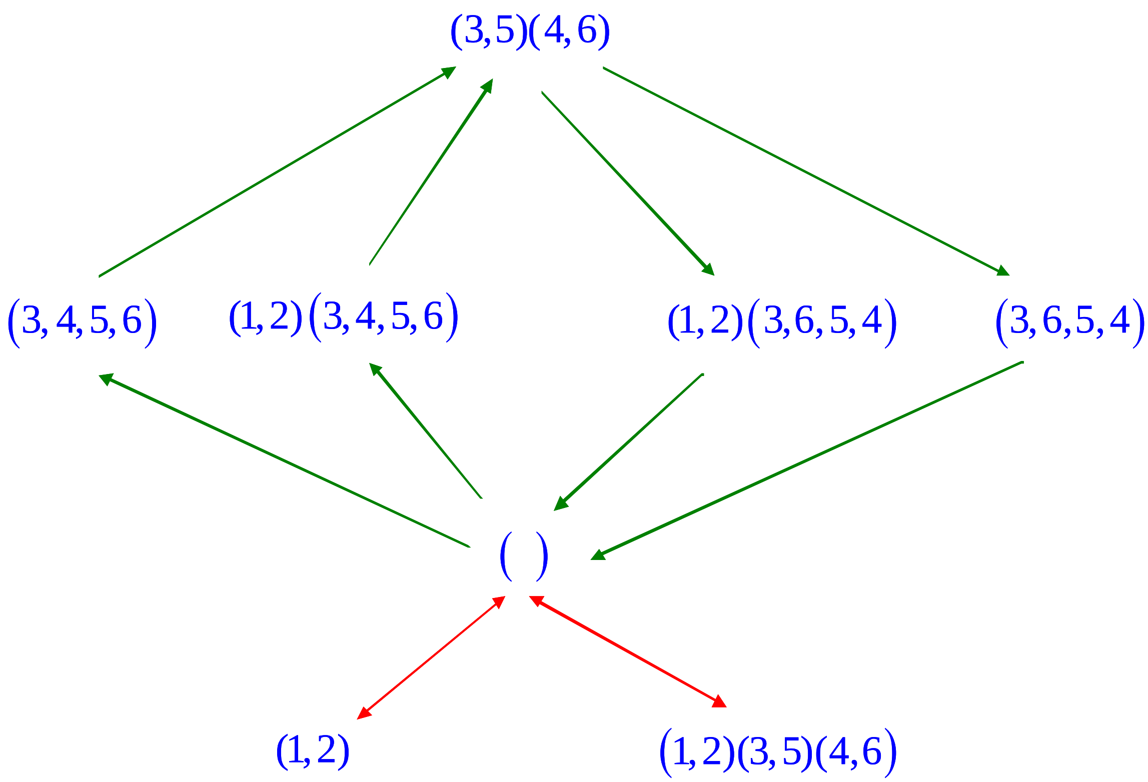
Elements:

$$\left\{\begin{matrix} (\ ) \\ (3,4,5,6) \\ (3,5)(4,6) \\ (3,6,5,4) \\ (1,2) \\ (1,2)(3,4,5,6) \\ (1,2)(3,5)(4,6) \\ (1,2)(3,6,5,4) \end{matrix}\right\} \cong C_2 \times C_4$$

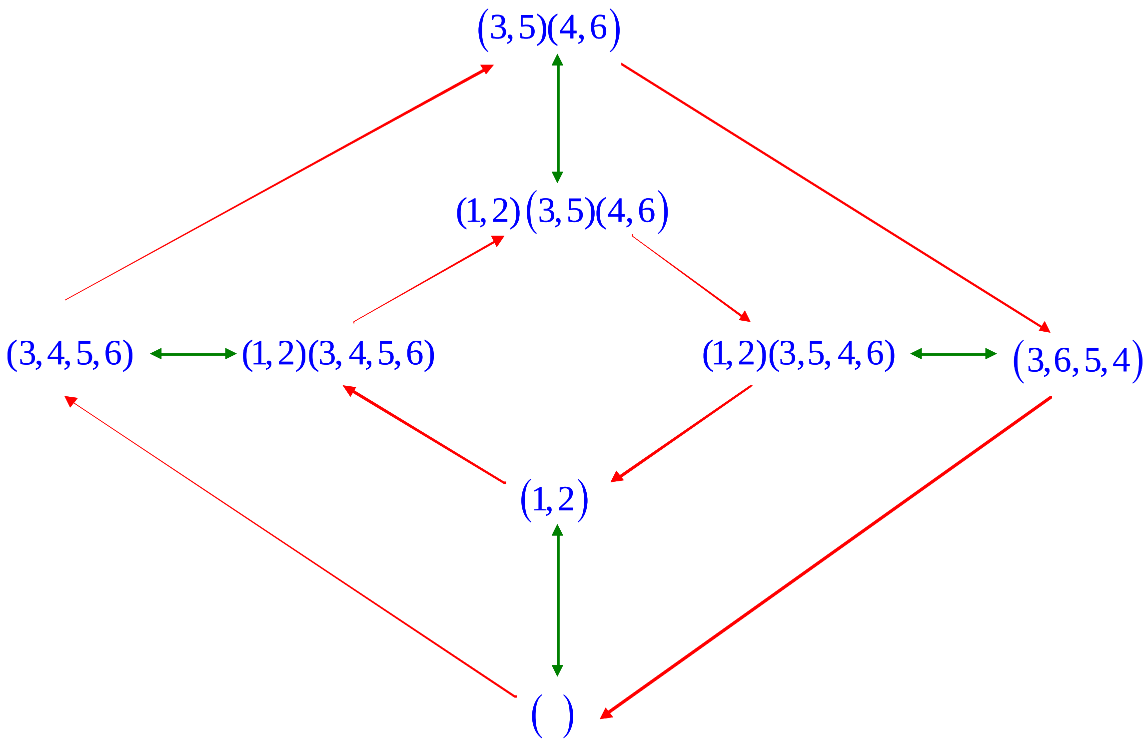
Is *Abelian*?

Yes

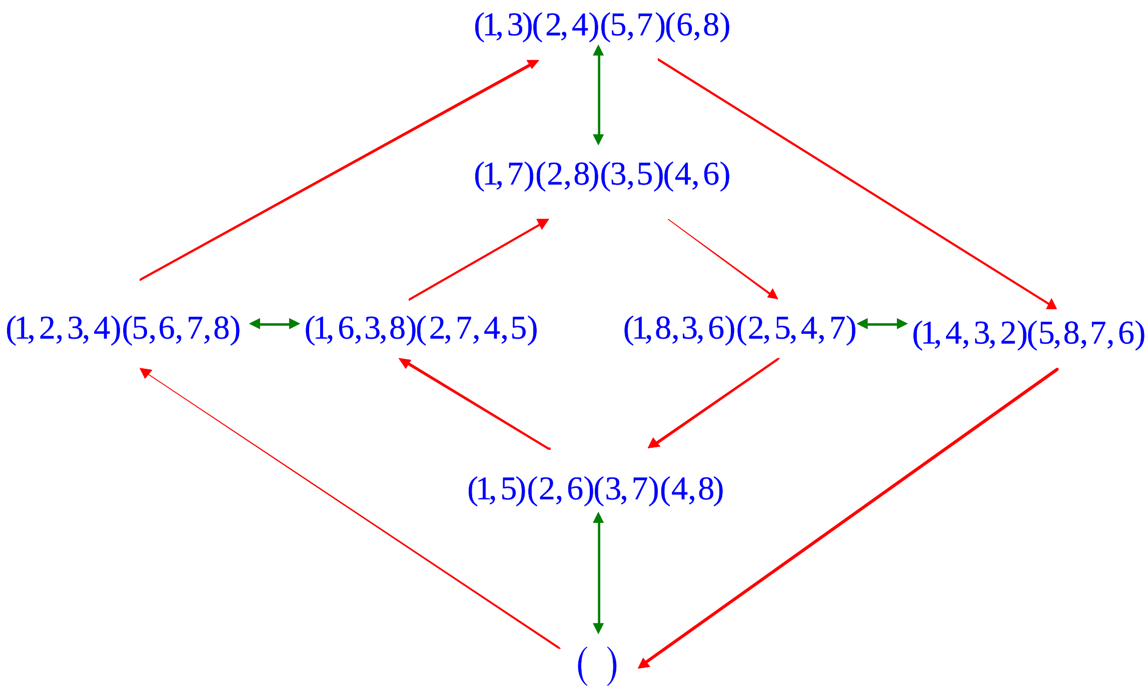
Cycle Graph



Cayley Diagram



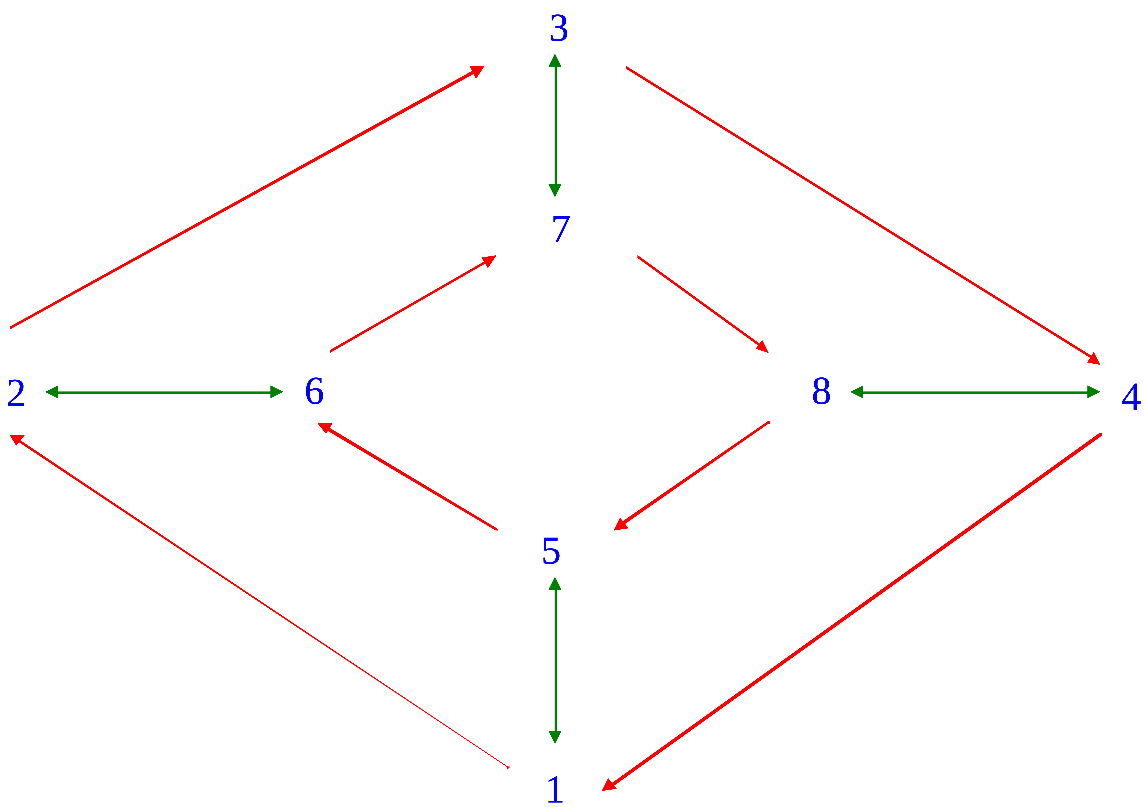
or



$a = (1,2,3,4)(5,6,7,8)$

$b = (1,5)(2,6)(3,7)(4,8)$

Generator Diagram



$a = (1,2,3,4)(5,6,7,8)$

$b = (1,5)(2,6)(3,7)(4,8)$

THE *DIRECT PRODUCT* $C_2 \times C_2 \times C_2$

$$C_2 \times C_2 \times C_2 \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$$

Generators:

$(1,2),(3,4),(5,6)$

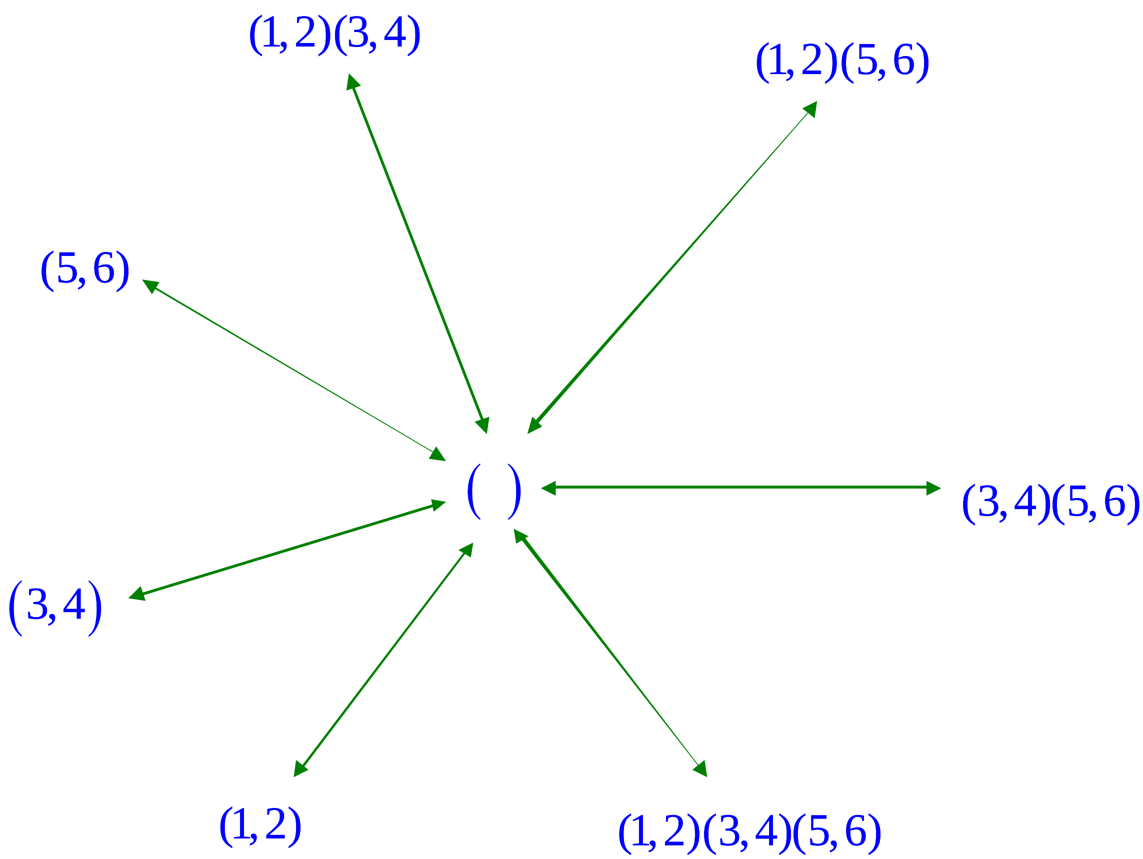
Elements:

$$\left\{ \begin{array}{c} (\ ) \\ (5,6) \\ (3,4) \\ (3,4)(5,6) \\ (1,2) \\ (1,2)(5,6) \\ (1,2)(3,4) \\ (1,2)(3,4)(5,6) \end{array} \right\} \cong C_2 \times C_2 \times C_2$$

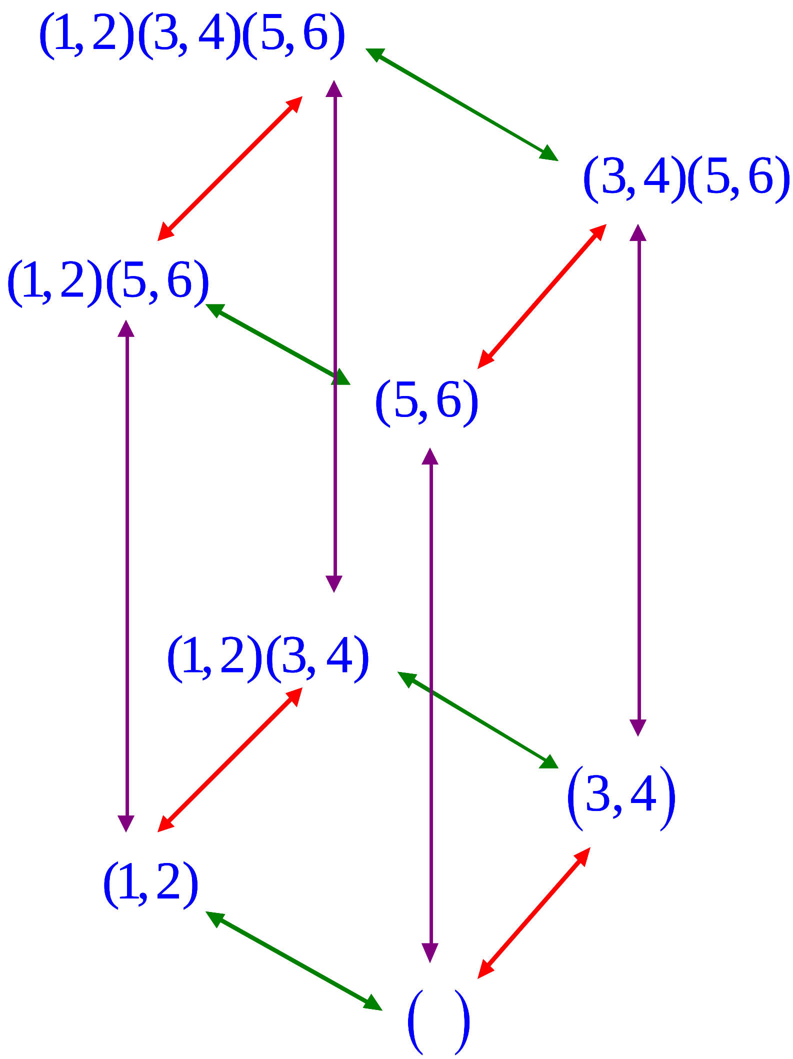
Is *Abelian*?

Yes

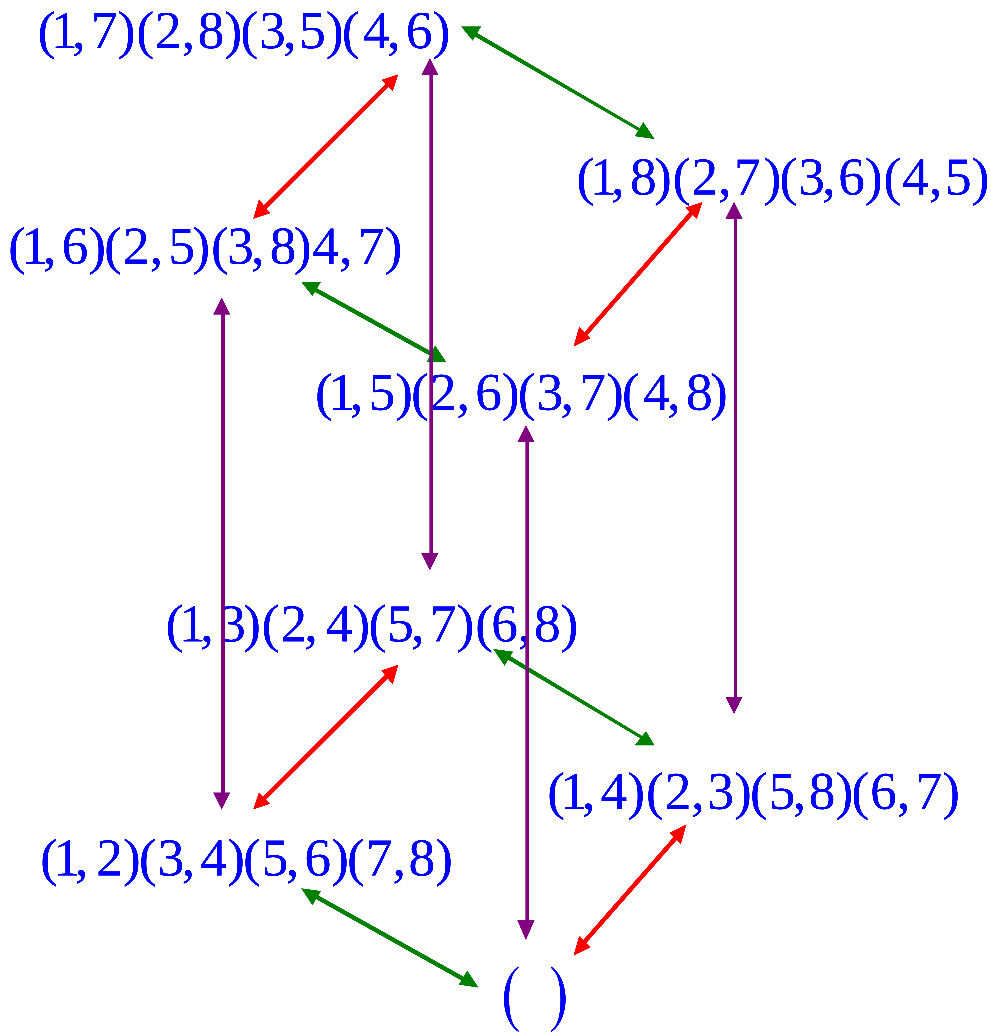
Cycle Graph



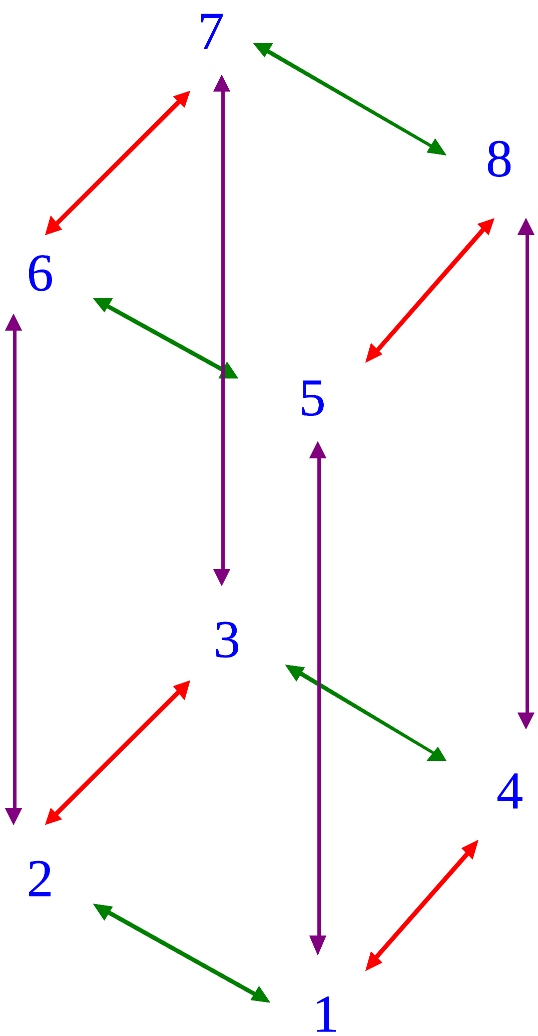
Cayley Diagram



or


(1,7)(2,8)(3,5)(4,6)
(1,8)(2,7)(3,6)(4,5)
(1,6)(2,5)(3,8)4,7)
(1,5)(2,6)(3,7)(4,8)
(1,3)(2,4)(5,7)(6,8)
(1,4)(2,3)(5,8)(6,7)
(1,2)(3,4)(5,6)(7,8)
( )

Generator Diagram



THE *DIHEDRAL GROUP* D_4

$$D_4 \cong C_4 \rtimes C_2 \cong \mathbb{Z}_4 \rtimes \mathbb{Z}_2$$

Generators:

$(1,2,3,4),(2,4)$

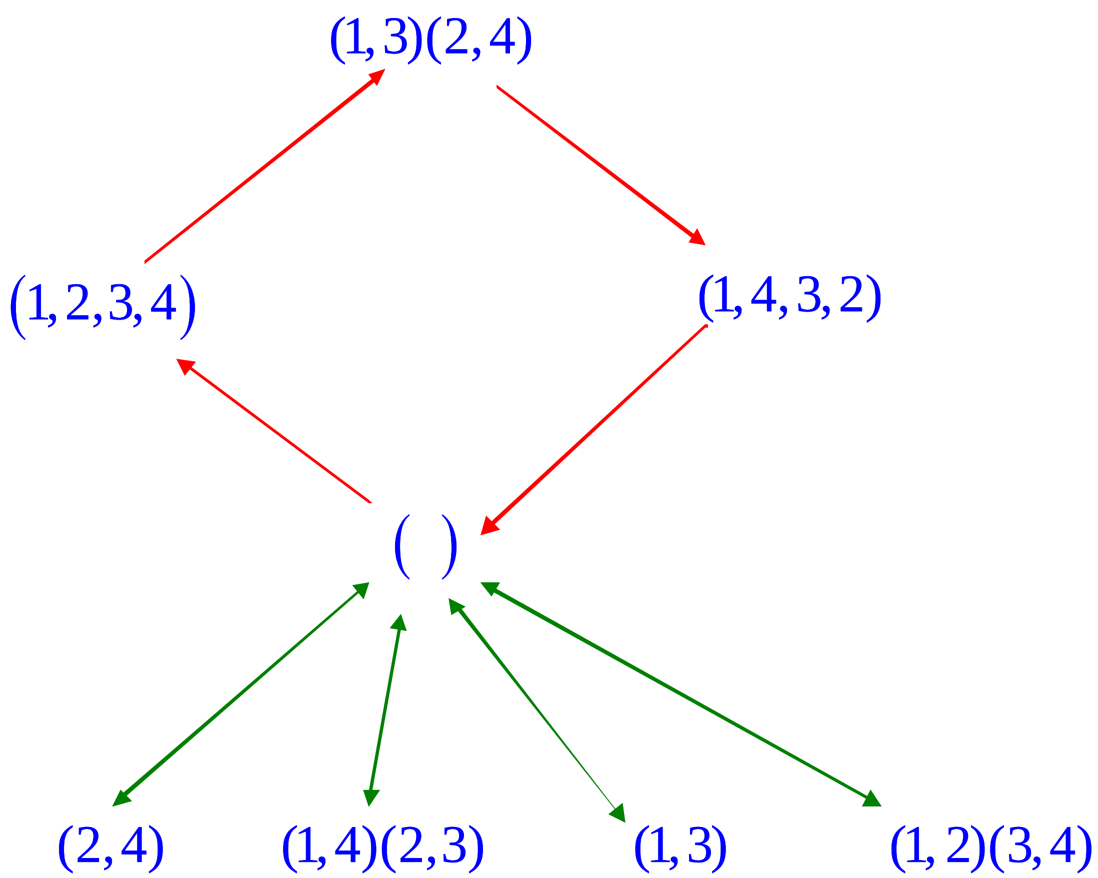
Elements:

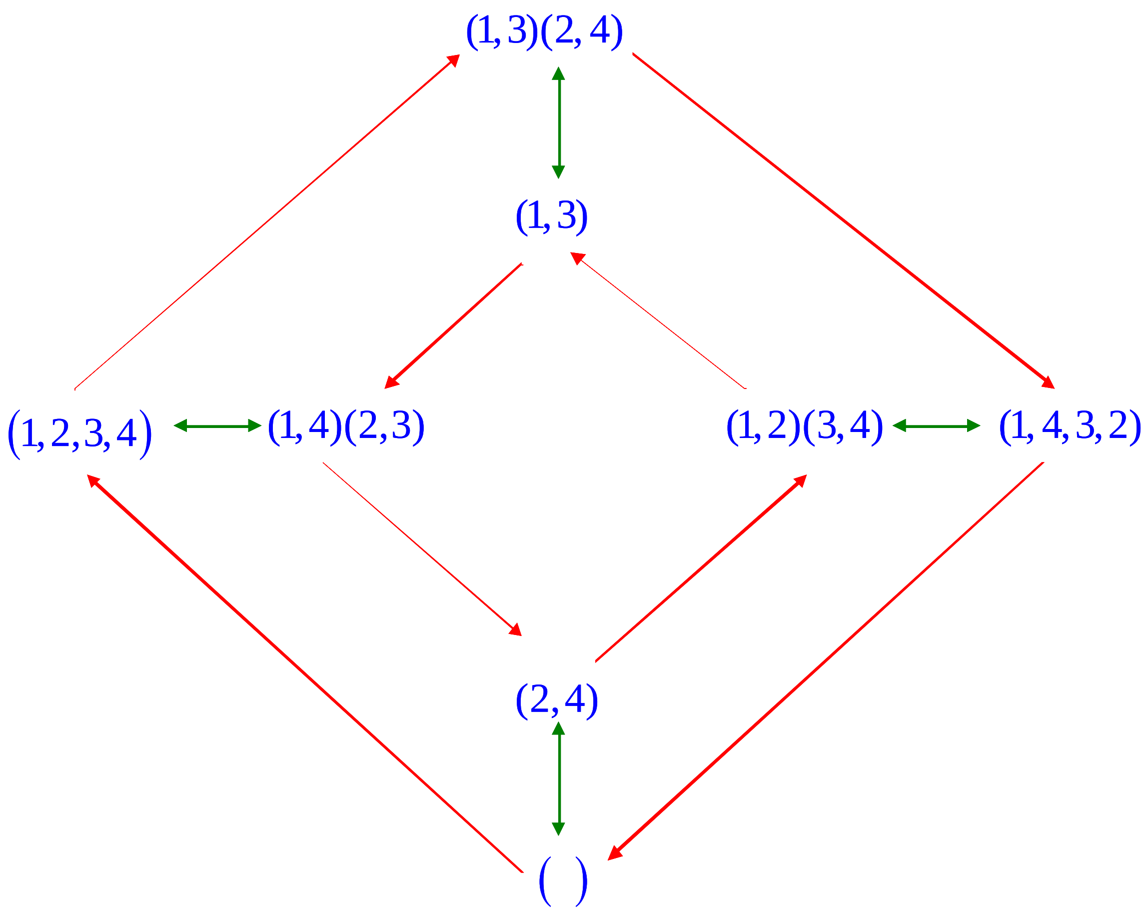
$$\left\{\begin{matrix} (\ ) \\ (2,4) \\ (1,2)(3,4) \\ (1,2,3,4) \\ (1,3) \\ (1,3)(2,4) \\ (1,4,3,2) \\ (1,4)(2,3) \end{matrix}\right\} \cong D_4$$

Is *Abelian*?

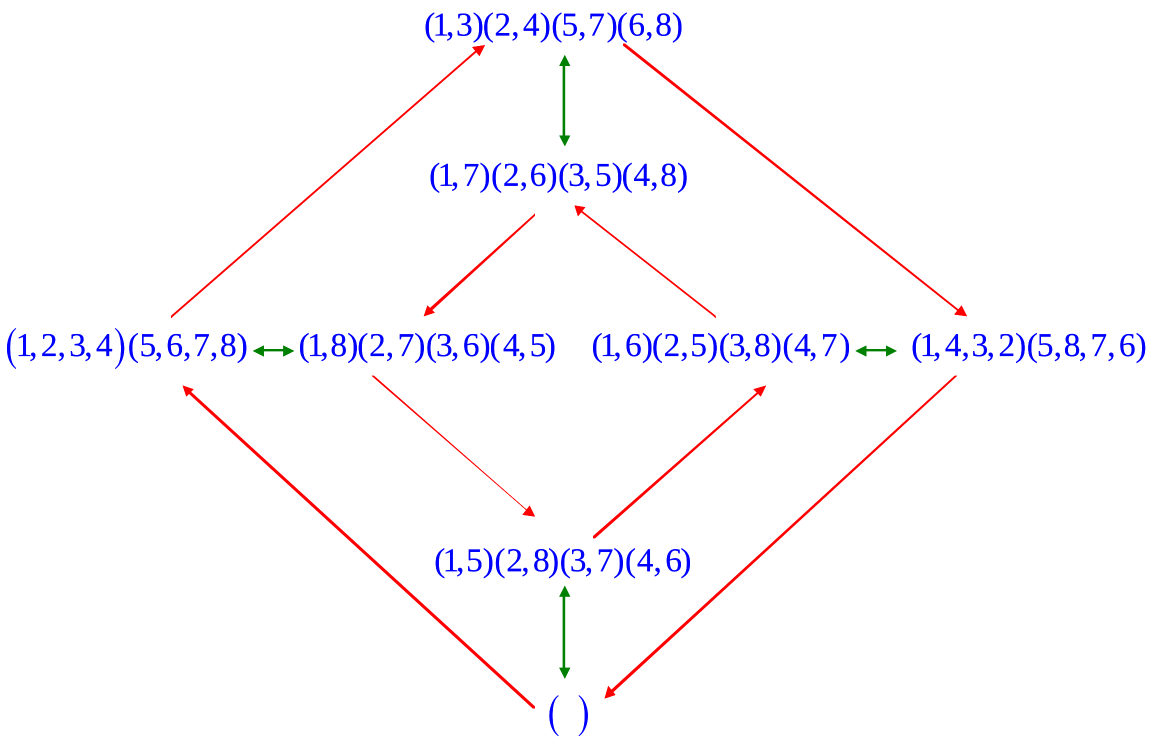
No

Cycle Graph



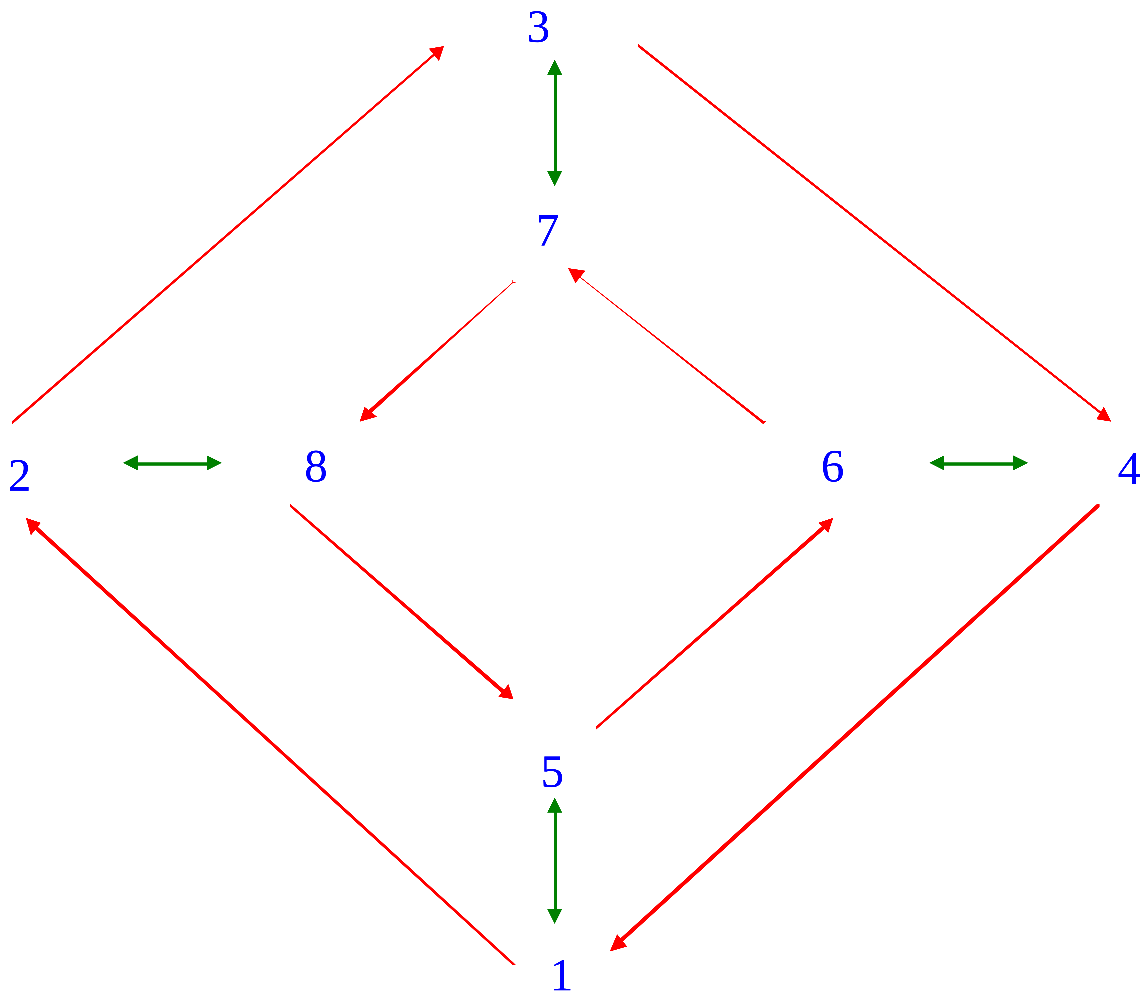


or



(1,3)(2,4)(5,7)(6,8)
(1,7)(2,6)(3,5)(4,8)
(1,2,3,4)(5,6,7,8)
(1,8)(2,7)(3,6)(4,5)
(1,6)(2,5)(3,8)(4,7)
(1,4,3,2)(5,8,7,6)
(1,5)(2,8)(3,7)(4,6)
( )

Generator Diagram



$a = (1,2,3,4)(5,6,7,8)$

$b = (1,5)(2,8)(3,7)(4,6)$

THE *QUATERNION GROUP* Q_8

Q_8

Generators:

$(1,2,5,6)(3,8,7,4)$, $(1,4,5,8)(2,7,6,3)$

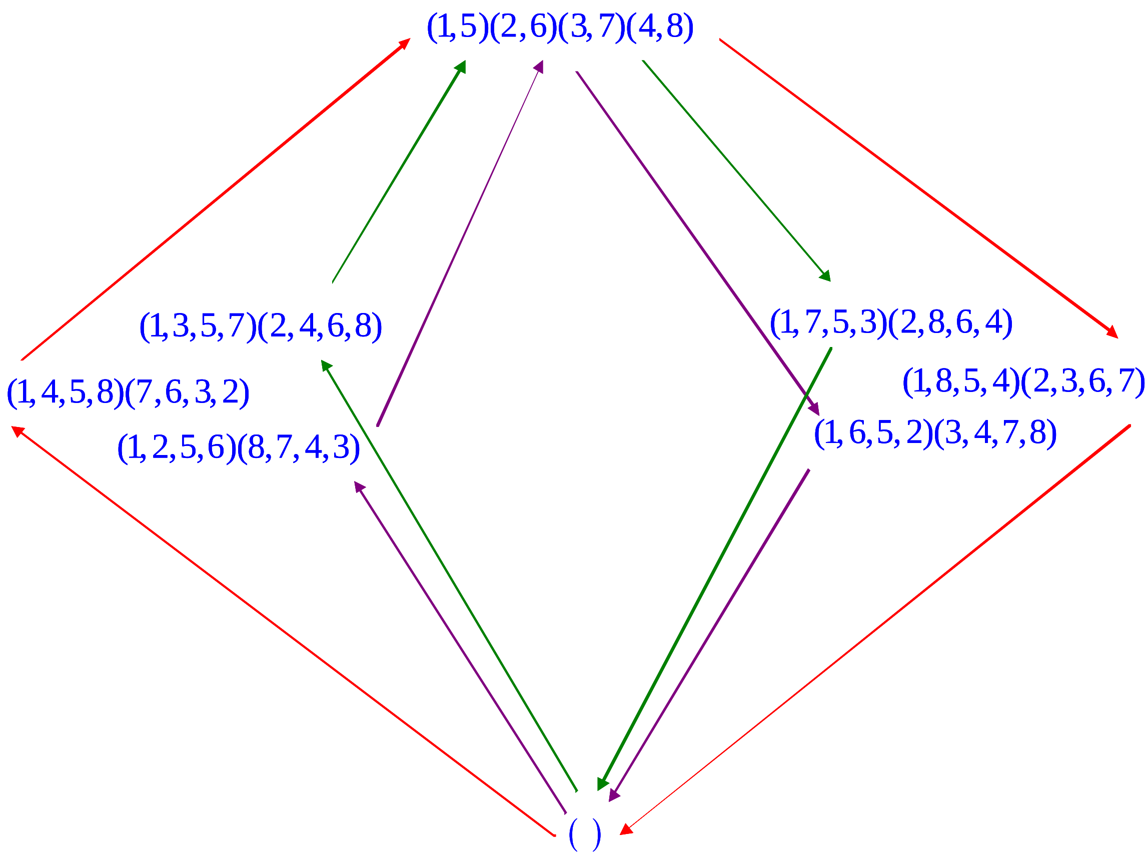
Elements:

$$\left\{\begin{matrix} (\ ) \\ (1,2,5,6)(3,8,7,4) \\ (1,3,5,7)(2,4,6,8) \\ (1,4,5,8)(2,7,6,3) \\ (1,5)(2,6)(3,7)(4,8) \\ (1,6,5,2)(3,4,7,8) \\ (1,7,5,3)(2,8,6,4) \\ (1,8,5,4)(2,3,6,7) \end{matrix}\right\} \cong Q_8$$

Is *Abelian*?

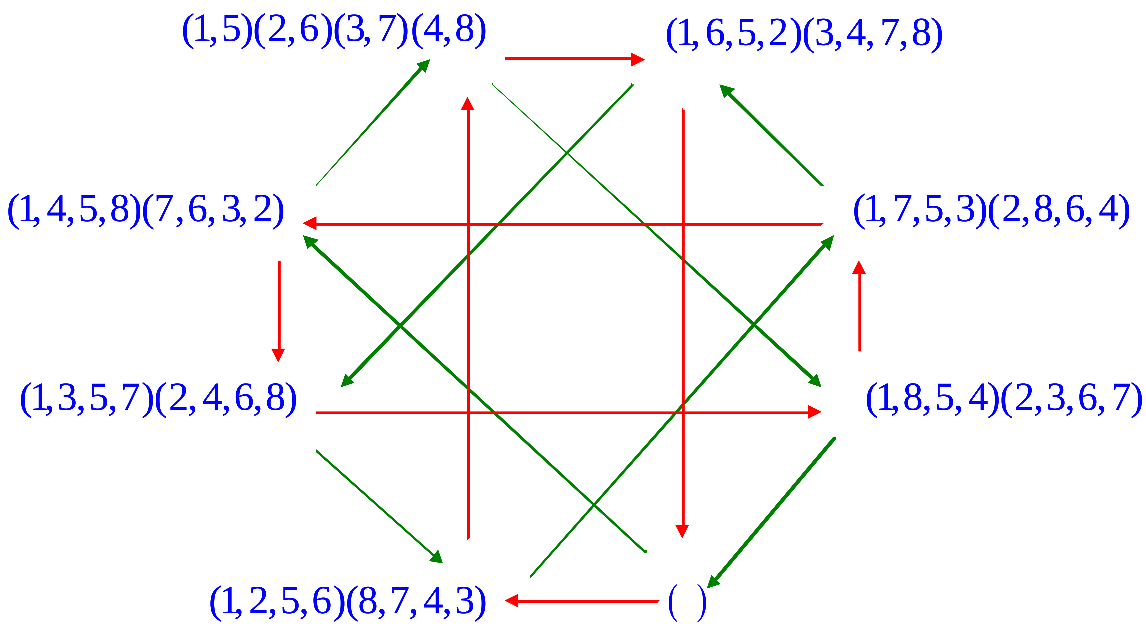
No

Cycle Graph

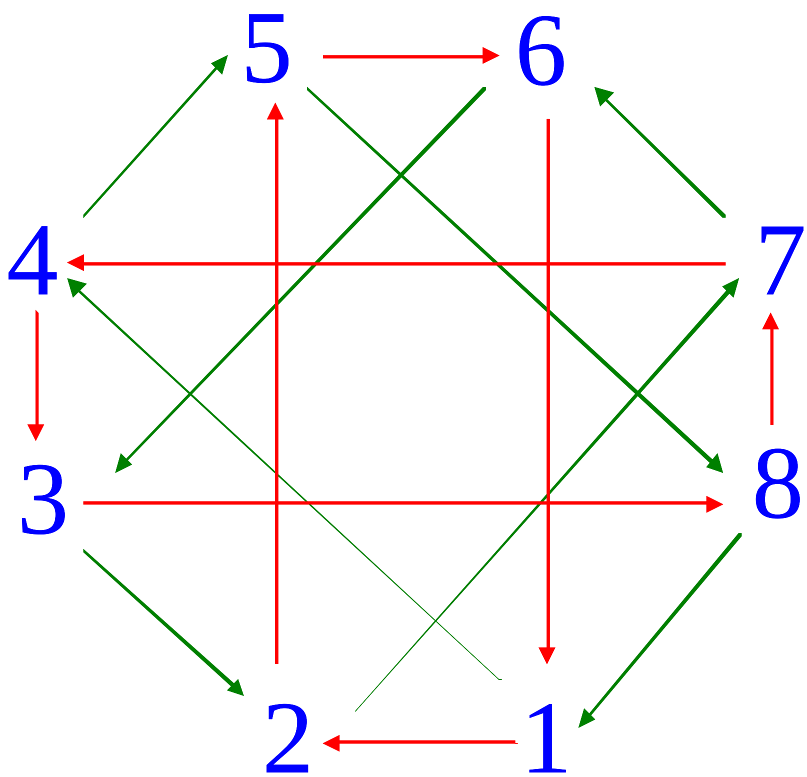


(1,5)(2,6)(3,7)(4,8)
(1,3,5,7)(2,4,6,8)
(1,4,5,8)(7,6,3,2)
(1,2,5,6)(8,7,4,3)
(1,7,5,3)(2,8,6,4)
(1,8,5,4)(2,3,6,7)
(1,6,5,2)(3,4,7,8)
( )

Cayley Diagram



Generator Diagram



Groups of Order 9

Things get simpler again once we get to order 9. There are only two *groups* of order 9, and they are both *abelian*. Thus, the only two possible *groups* of this order are C_9 and $C_3 \times C_3$.

THE *CYCLIC GROUP* OF ORDER 9

$$C_9 \cong \mathbb{Z}_9$$

Generators:

$(1,2,3,4,5,6,7,8,9)$

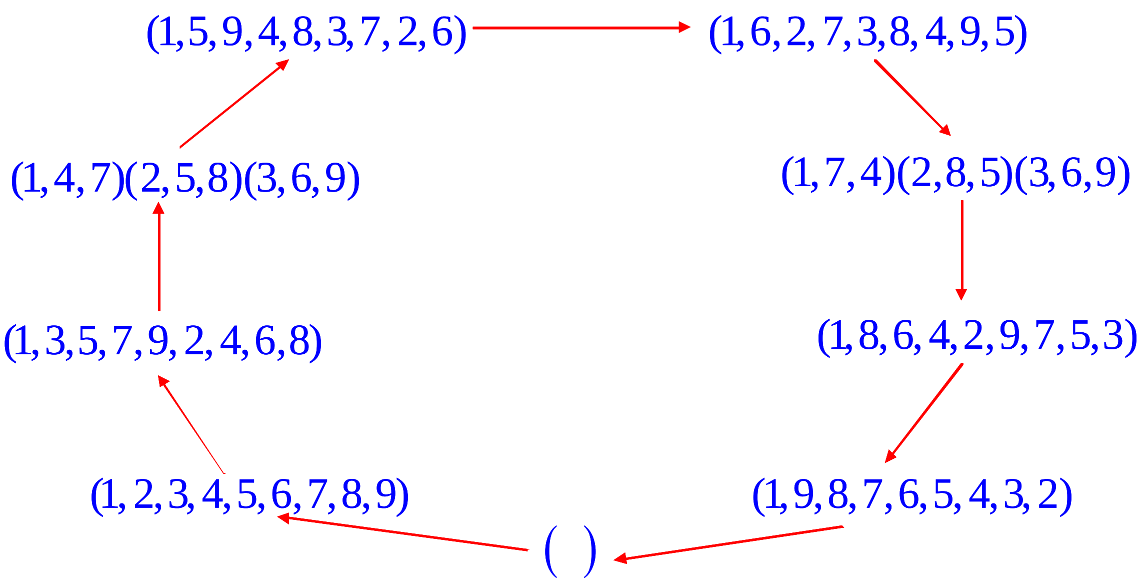
Elements:

$$\left\{\begin{array}{c} (\ ) \\ (1,2,3,4,5,6,7,8,9) \\ (1,3,5,7,9,2,4,6,8) \\ (1,4,7)(2,5,8)(3,6,9) \\ (1,5,9,4,8,3,7,2,6) \\ (1,6,2,7,3,8,4,9,5) \\ (1,7,4)(2,8,5)(3,6,9) \\ (1,8,6,4,2,9,7,5,3) \\ (1,9,8,7,6,5,4,3,2 \end{array}\right\} \cong C_9$$

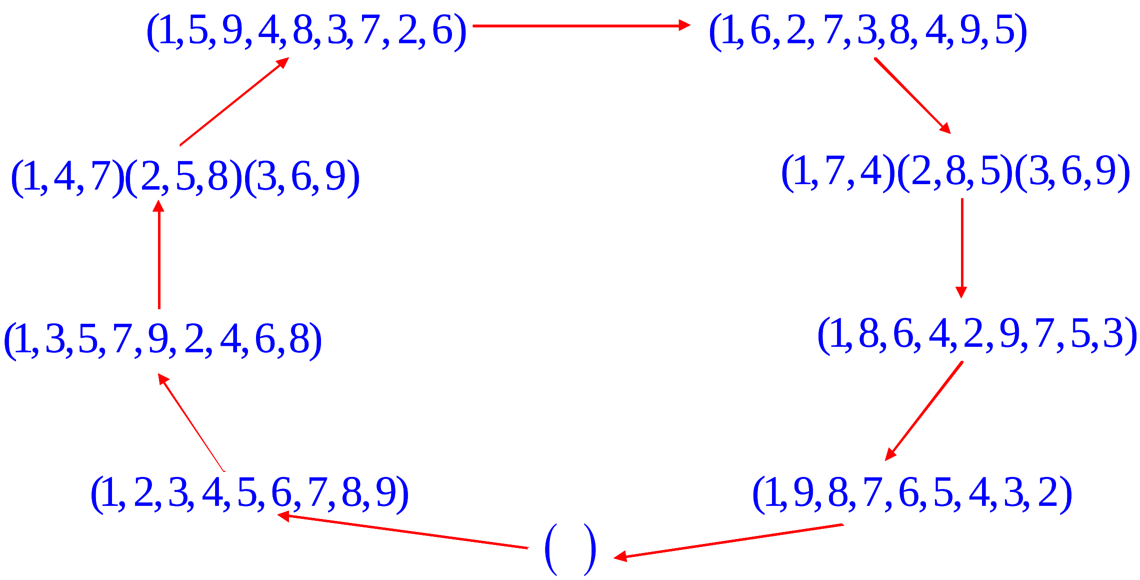
Is *Abelian*?

Yes

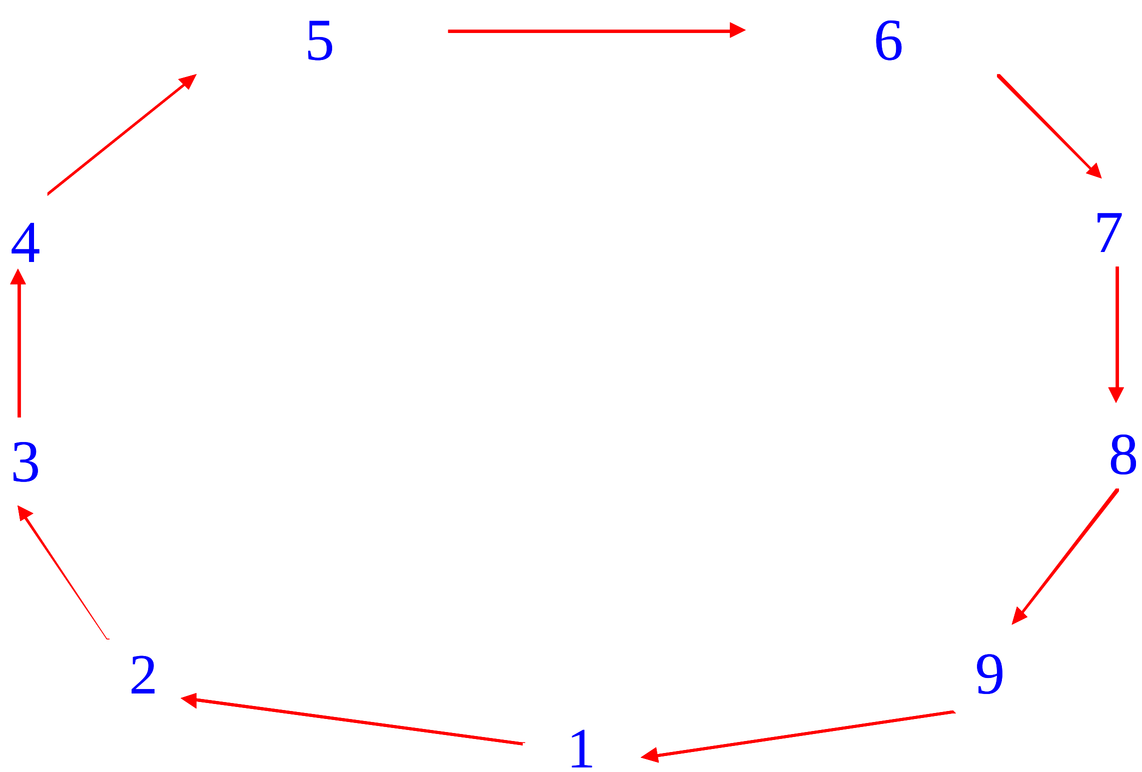
Cycle Graph


(1,5,9,4,8,3,7,2,6)
(1,6,2,7,3,8,4,9,5)
(1,4,7)(2,5,8)(3,6,9)
(1,7,4)(2,8,5)(3,6,9)
(1,3,5,7,9,2,4,6,8)
(1,8,6,4,2,9,7,5,3)
(1,2,3,4,5,6,7,8,9)
(1,9,8,7,6,5,4,3,2)
( )

Cayley Diagram



Generator Diagram



THE *DIRECT PRODUCT* $\mathbb{Z}_3 \times \mathbb{Z}_3$

$$\mathbb{Z}_3 \times \mathbb{Z}_3 \cong C_3 \times C_3$$

Generators:

$(1,2,3),(4,5,6)$

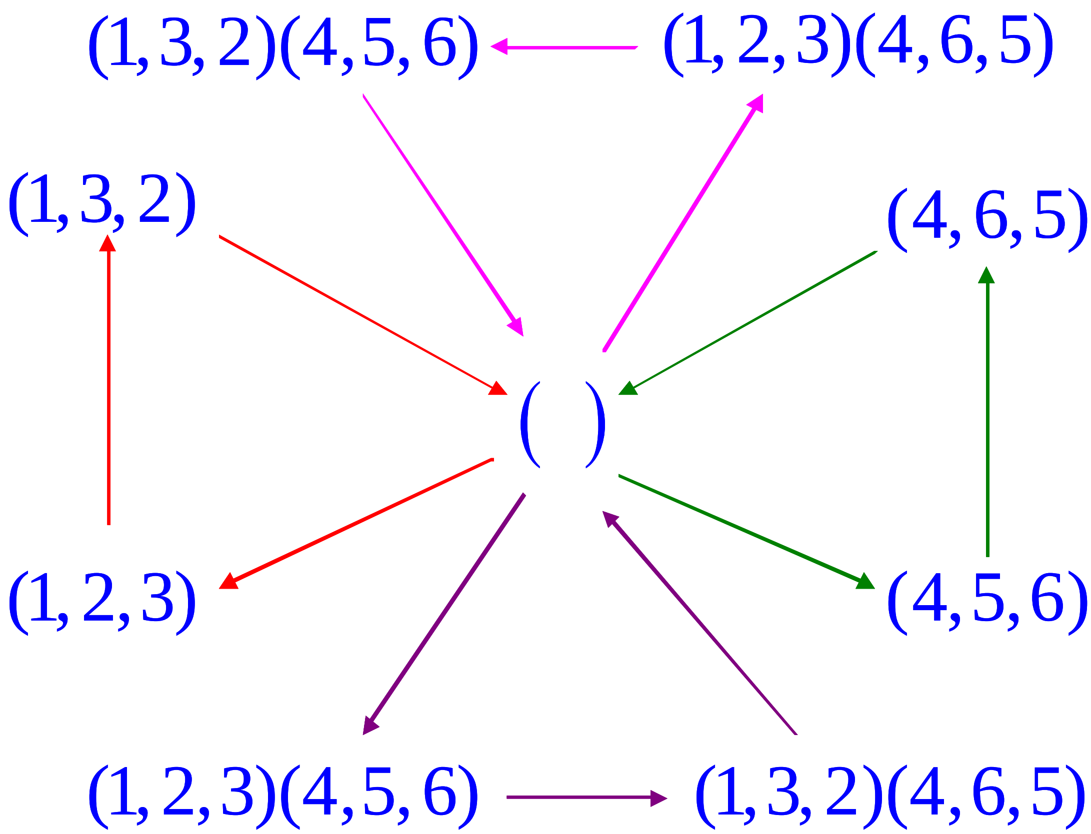
Elements:

$$\left\{\begin{matrix} (\ ) \\ (4,5,6) \\ (4,6,5) \\ (1,2,3) \\ (1,2,3)(4,5,6) \\ (1,2,3)(4,6,5) \\ (1,3,2) \\ (1,3,2)(4,5,6) \\ (1,3,2)(4,6,5) \end{matrix}\right\} \cong C_3 \times C_3$$

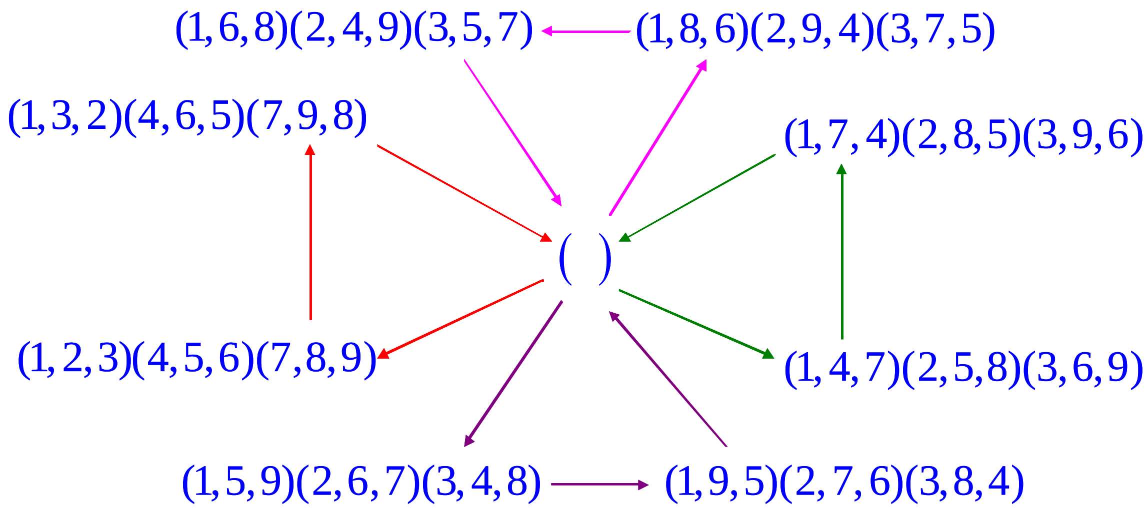
Is *Abelian*?

Yes

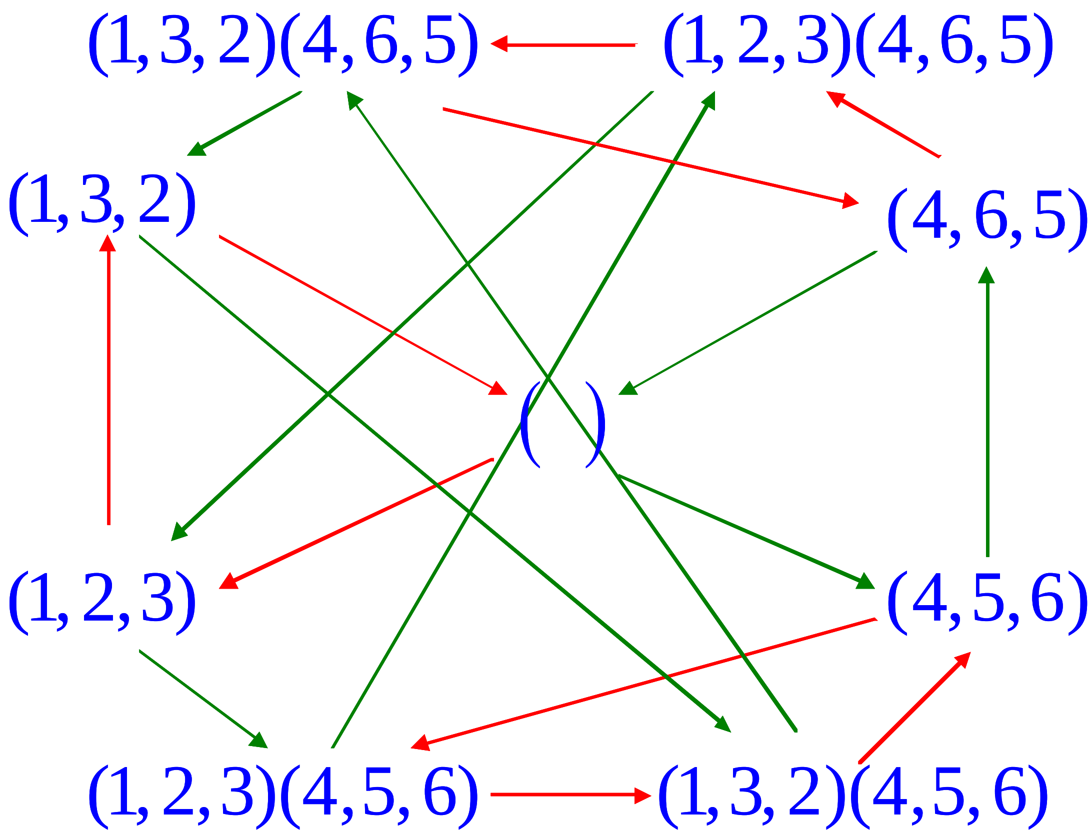
Cycle Graph



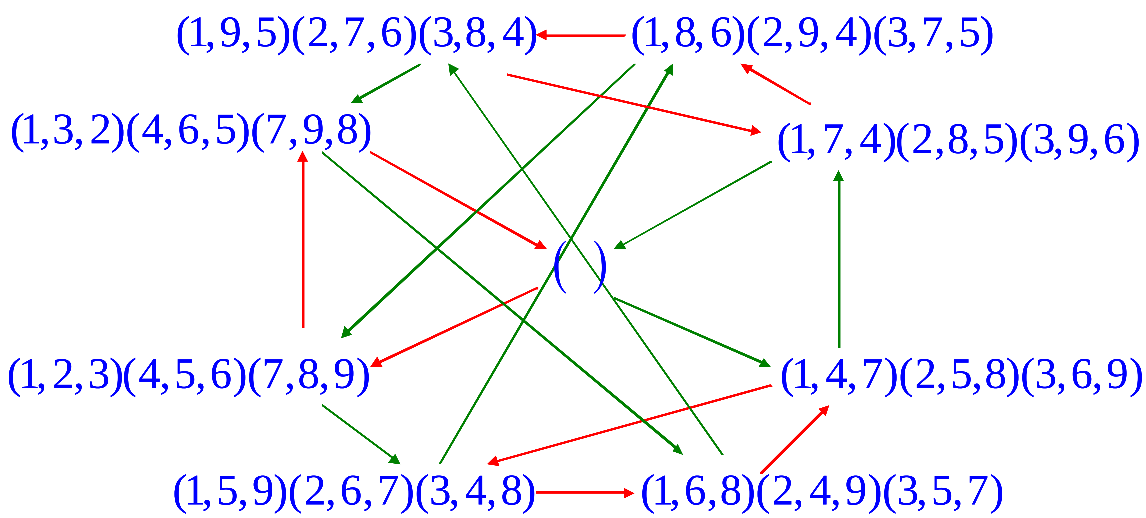
or


(1,6,8)(2,4,9)(3,5,7)
(1,8,6)(2,9,4)(3,7,5)
(1,3,2)(4,6,5)(7,9,8)
(1,7,4)(2,8,5)(3,9,6)
( )
(1,2,3)(4,5,6)(7,8,9)
(1,4,7)(2,5,8)(3,6,9)
(1,5,9)(2,6,7)(3,4,8)
(1,9,5)(2,7,6)(3,8,4)

Cayley Diagram

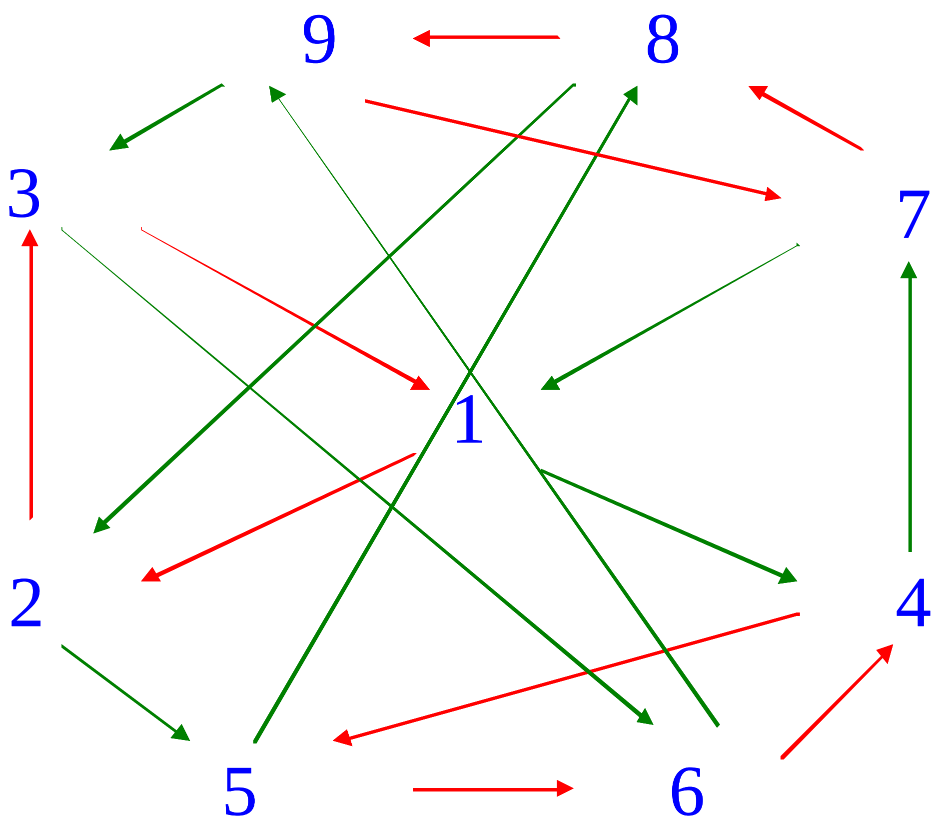


or



(1,9,5)(2,7,6)(3,8,4)
(1,8,6)(2,9,4)(3,7,5)
(1,3,2)(4,6,5)(7,9,8)
(1,7,4)(2,8,5)(3,9,6)
( )
(1,2,3)(4,5,6)(7,8,9)
(1,4,7)(2,5,8)(3,6,9)
(1,5,9)(2,6,7)(3,4,8)
(1,6,8)(2,4,9)(3,5,7)

Generator Diagram



GROUPS OF ORDER 10

Things are also pretty simple for *groups* of order 10. We know that one *group* of order 10 is the *abelian group* $C_{10} \cong C_5 \times C_2$, and the other is the *nonabelian group* $D_5 \cong C_5 >\!\!\lhd C_2$.

THE *CYCLIC GROUP* OF ORDER 10

$$C_{10} \cong C_2 \times C_5 \cong \mathbb{Z}_2 \times \mathbb{Z}_5 \cong \mathbb{Z}_{10}$$

Generators:

$(1,2,3,4,5,6,7,8,9,10)$

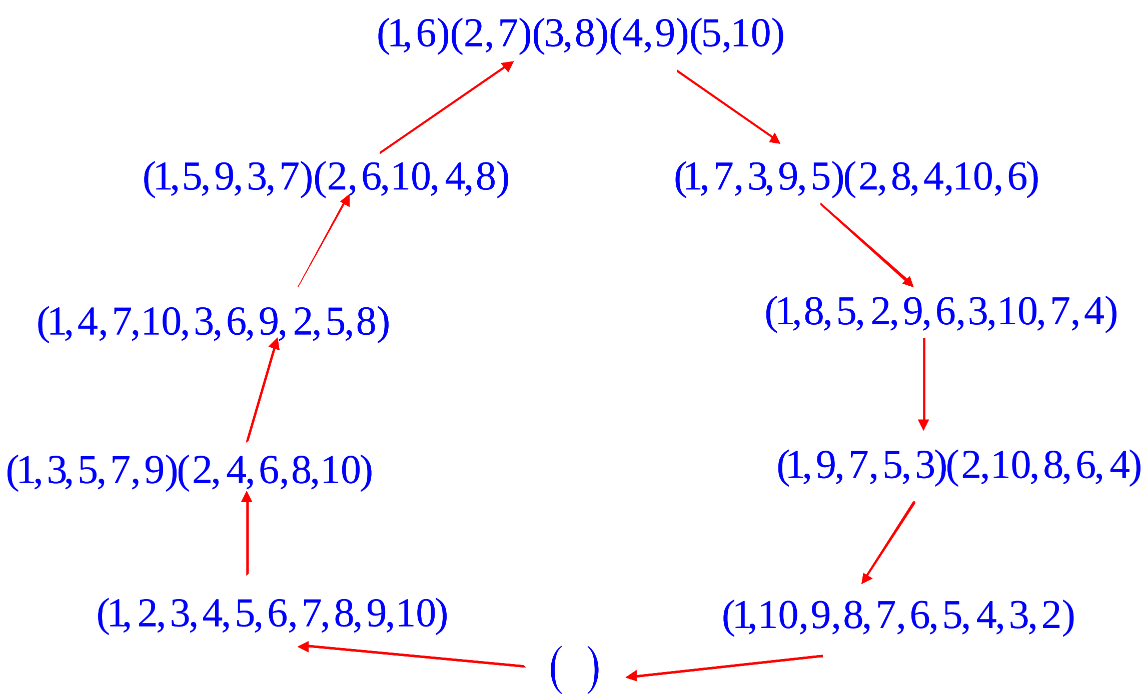
Elements:

$$\left\{\begin{array}{c} (\ ) \\ (1,2,3,4,5,6,7,8,9,10) \\ (1,3,5,7,9)(2,4,6,8,10) \\ (1,4,7,10,3,6,9,2,5,8) \\ (1,5,9,3,7)(2,6,10,4,8) \\ (1,6)(2,7)(3,8)(4,9)(5,10) \\ (1,7,3,9,5)(2,8,4,10,6) \\ (1,8,5,2,9,6,3,10,7,4) \\ (1,9,7,5,3)(2,10,8,6,4) \\ (1,10,9,8,7,6,5,4,3,2) \end{array}\right\} \cong C_{10}$$

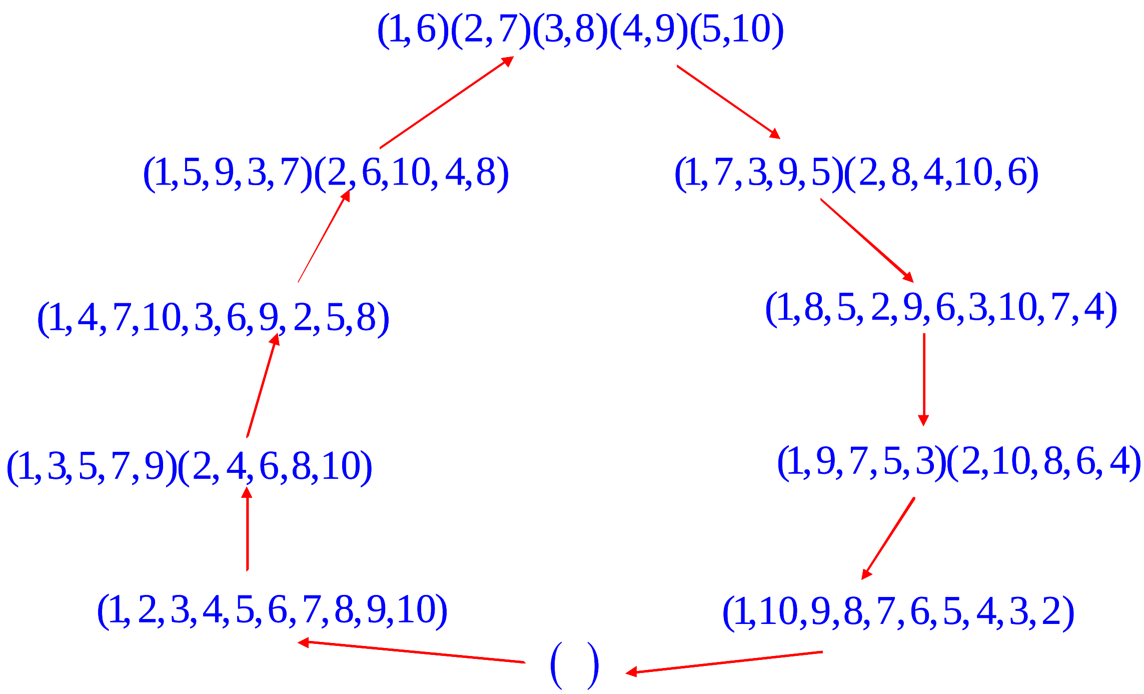
Is *Abelian*?

Yes

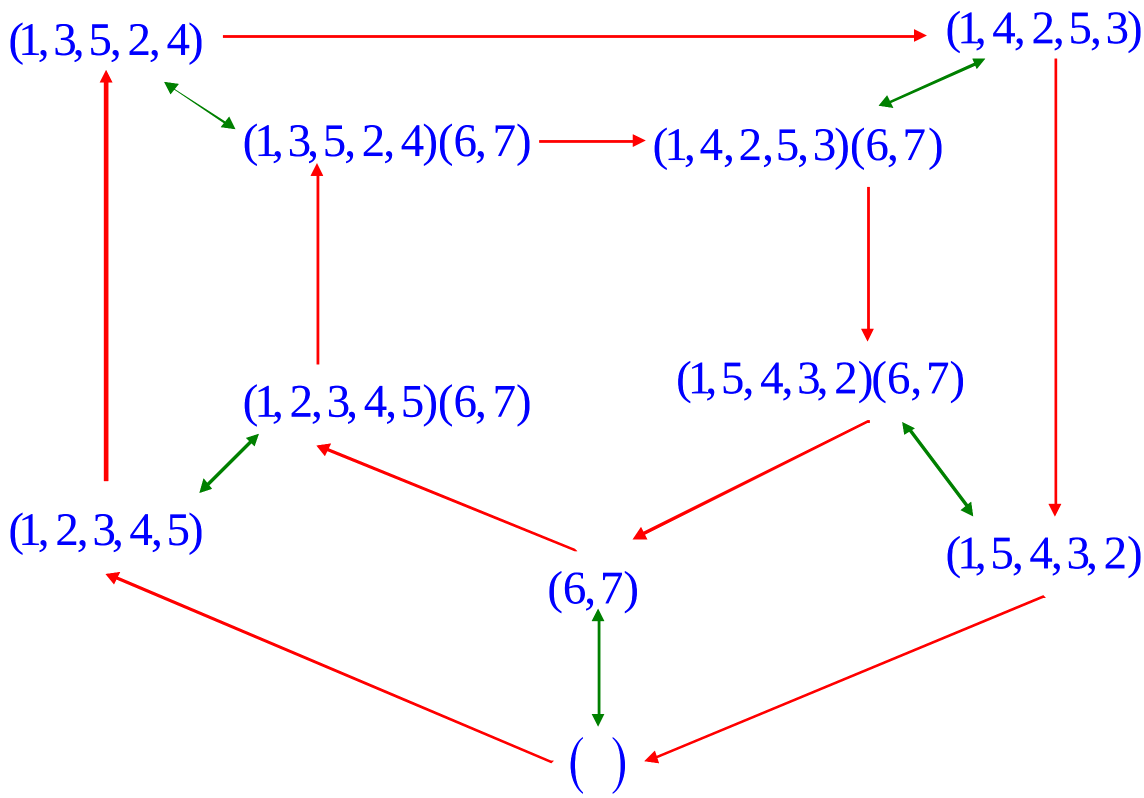
Cycle Graph


(1,6)(2,7)(3,8)(4,9)(5,10)
(1,5,9,3,7)(2,6,10,4,8)
(1,7,3,9,5)(2,8,4,10,6)
(1,4,7,10,3,6,9,2,5,8)
(1,8,5,2,9,6,3,10,7,4)
(1,3,5,7,9)(2,4,6,8,10)
(1,9,7,5,3)(2,10,8,6,4)
(1,2,3,4,5,6,7,8,9,10)
(1,10,9,8,7,6,5,4,3,2)
( )

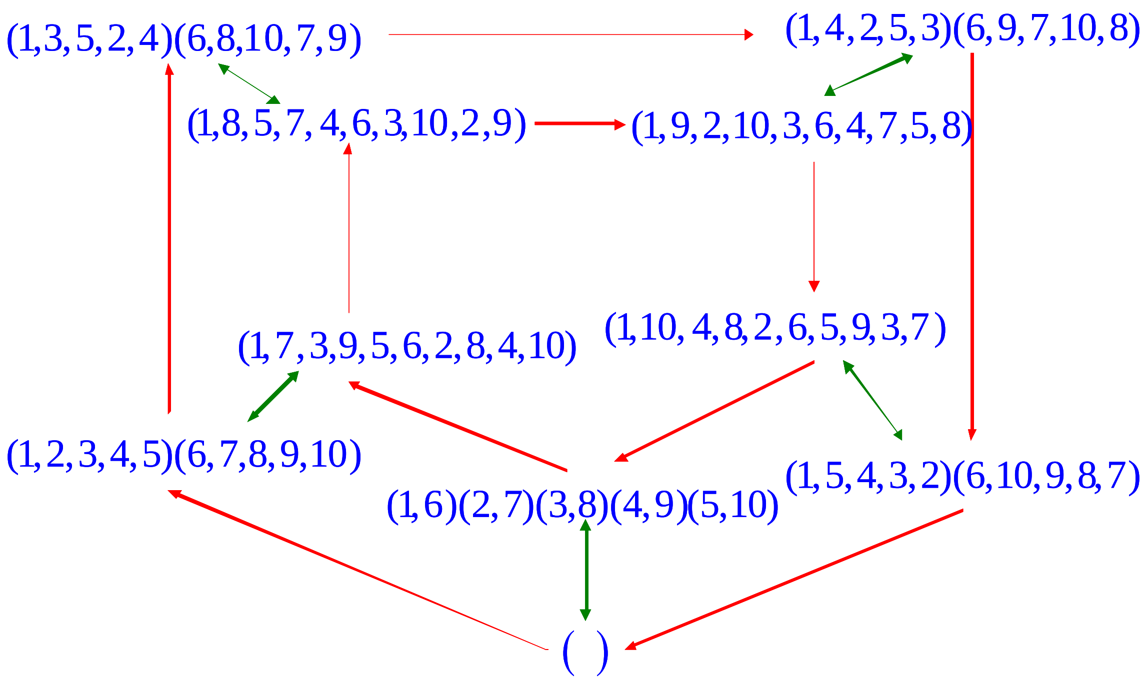
Cayley Diagram



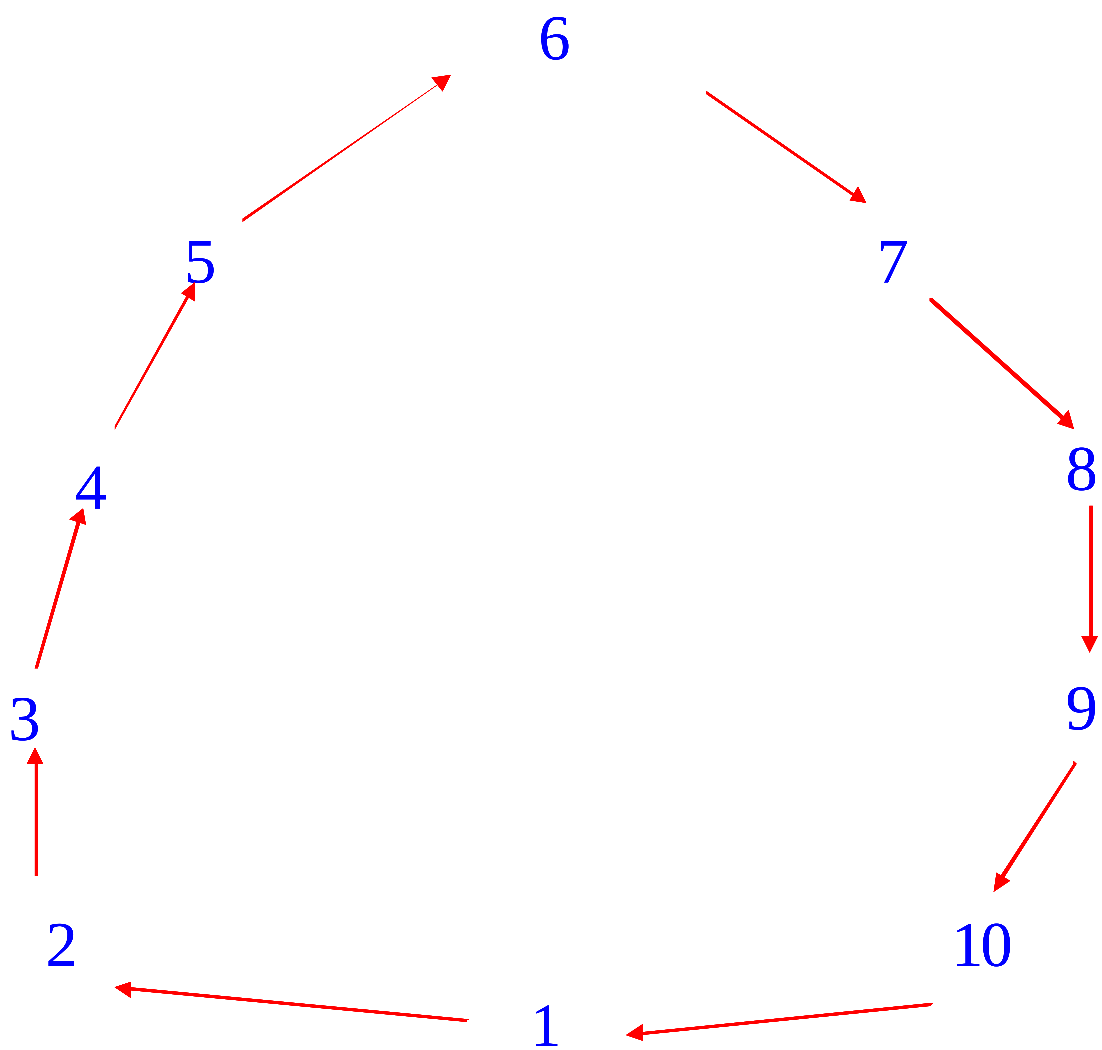
or


(1,3,5,2,4)
(1,4,2,5,3)
(1,3,5,2,4)(6,7)
(1,4,2,5,3)(6,7)
(1,5,4,3,2)(6,7)
(1,2,3,4,5)(6,7)
(1,2,3,4,5)
(1,5,4,3,2)
(6,7)
( )

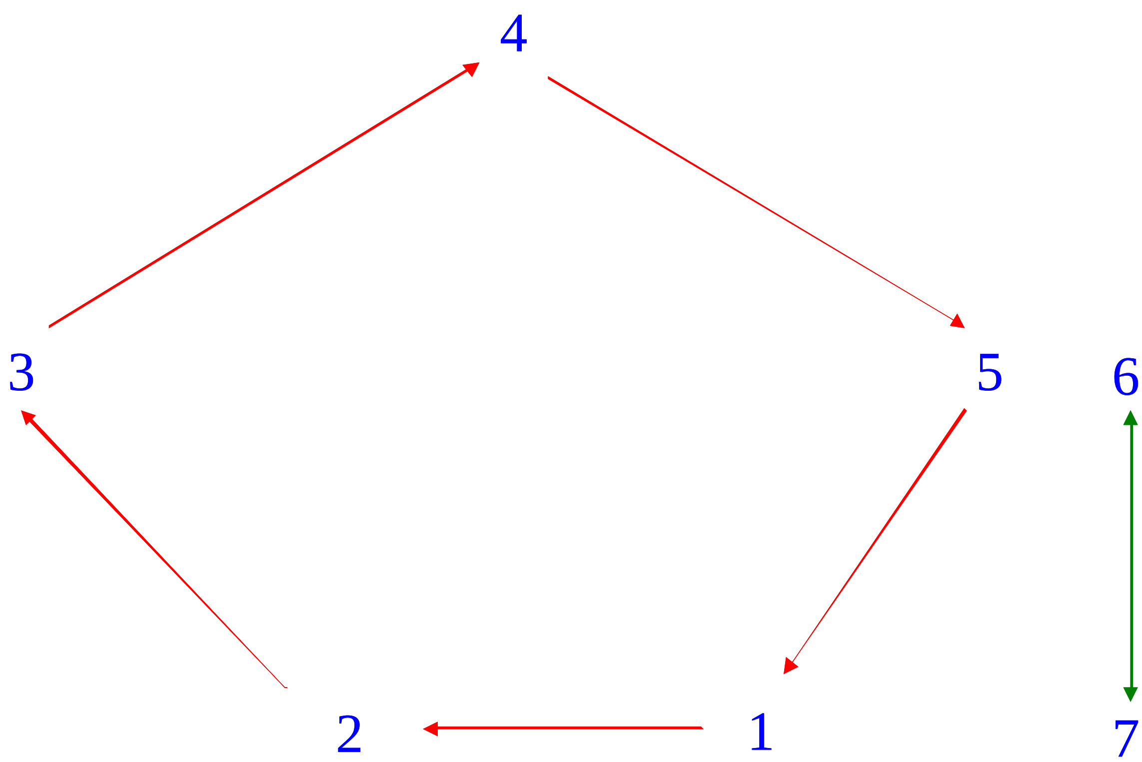
or


(1,3,5,2,4)(6,8,10,7,9)
(1,4,2,5,3)(6,9,7,10,8)
(1,8,5,7,4,6,3,10,2,9)
(1,9,2,10,3,6,4,7,5,8)
(1,7,3,9,5,6,2,8,4,10)
(1,10,4,8,2,6,5,9,3,7)
(1,2,3,4,5)(6,7,8,9,10)
(1,5,4,3,2)(6,10,9,8,7)
(1,6)(2,7)(3,8)(4,9)(5,10)
( )

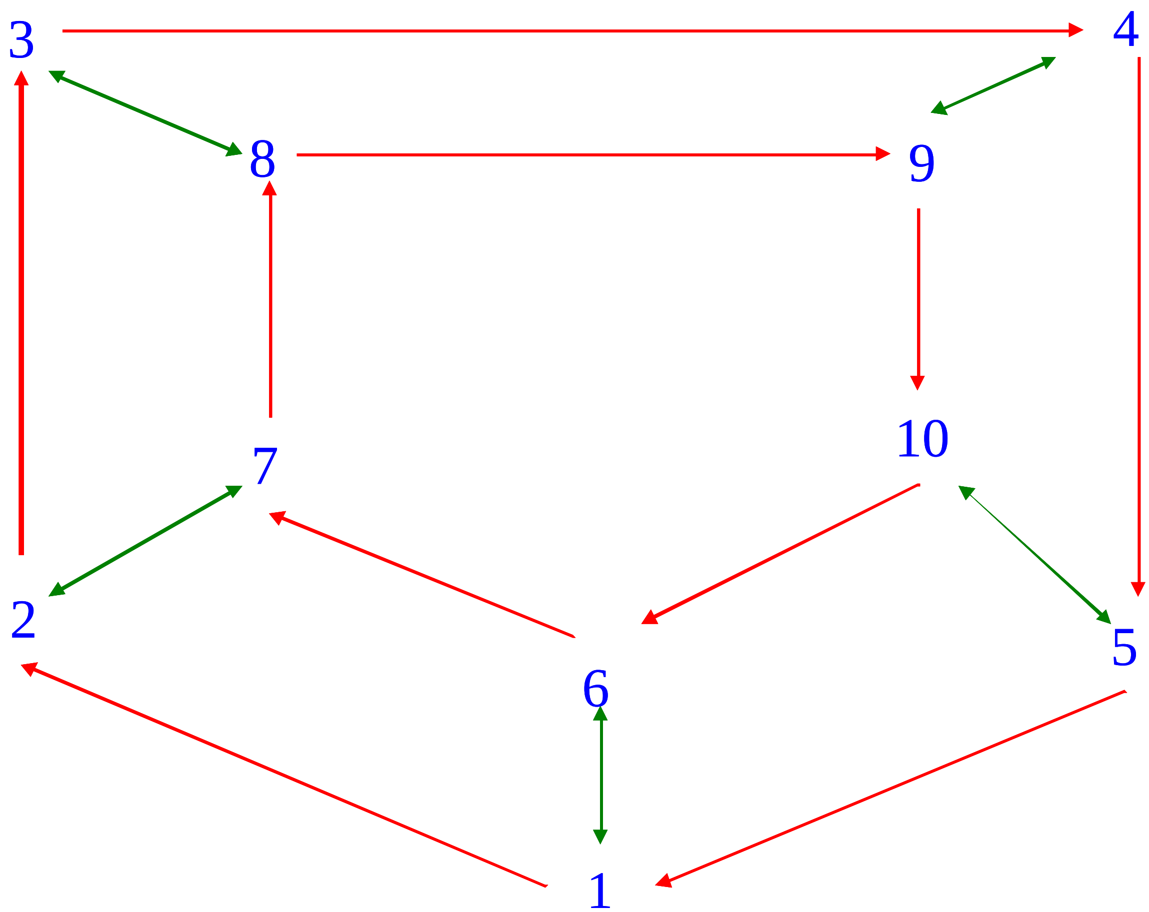
Generator Diagram



or



or


3
4
8
9
10
7
2
6
5
1

THE *DIHEDRAL GROUP* D_5

$$D_5 \cong \mathbb{Z}_5 >\!\!\lhd \mathbb{Z}_2 \cong C_5 >\!\!\lhd C_2$$

Generators:

$(1,2,3,4,5),(2,5)(3,4)$

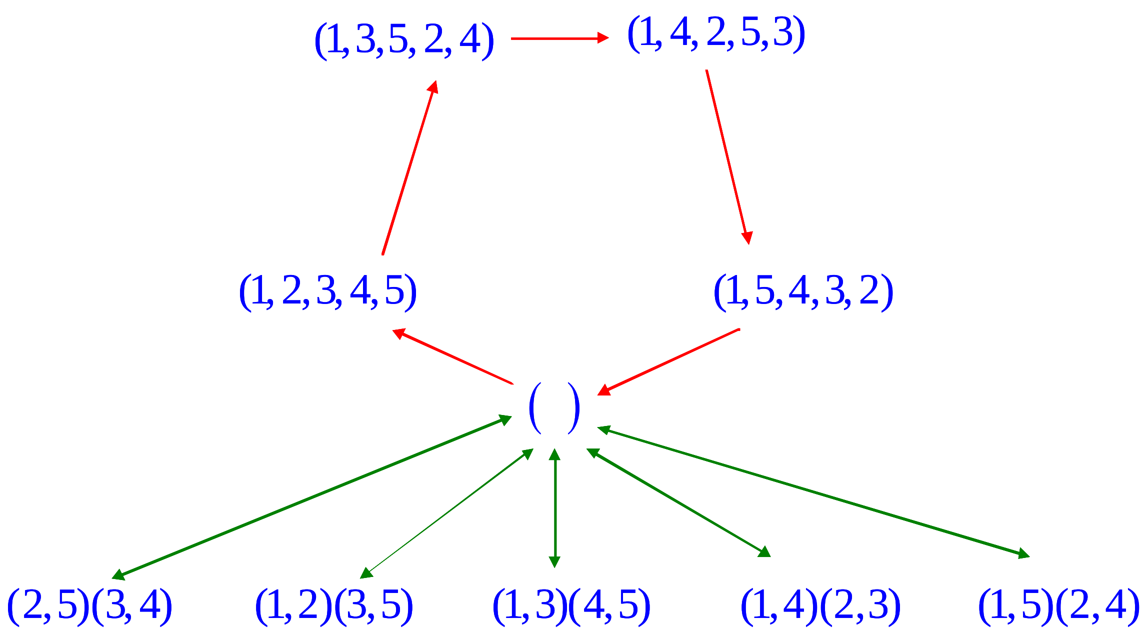
Elements:

$$\left\{\begin{matrix} (\ ) \\ (2,5)(3,4) \\ (1,2)(3,5) \\ (1,2,3,4,5) \\ (1,3)(4,5) \\ (1,3,5,2,4) \\ (1,4)(2,3) \\ (1,4,2,5,3) \\ (1,5,4,3,2) \\ (1,5)(2,4) \end{matrix}\right\} \cong \left\{\begin{matrix} (\ ) \\ (1,2,3,4,5)(6,7,8,9,10) \\ (1,3,5,2,4)(6,8,10,7,9) \\ (1,4,2,5,3)(6,9,7,10,8) \\ (1,5,4,3,2)(6,10,9,8,7) \\ (1,6)(2,10)(3,9)(4,8)(5,7) \\ (1,7)(2,6)(3,10)(4,9)(5,8) \\ (1,8)(2,7)(3,6)(4,10)(5,9) \\ (1,9)(2,8)(3,7)(4,6)(5,10) \\ (1,10)(2,9)(3,8)(4,7)(5,6) \end{matrix}\right\} \cong D_5$$

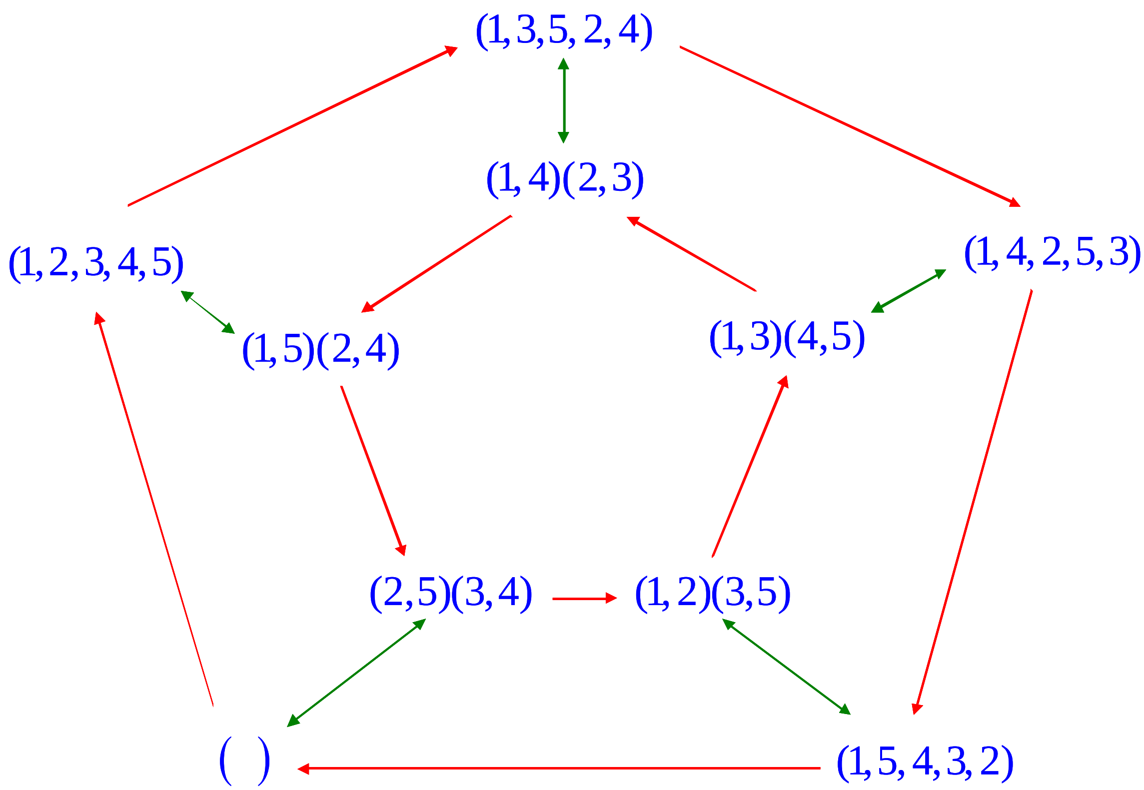
Is *Abelian*?

No

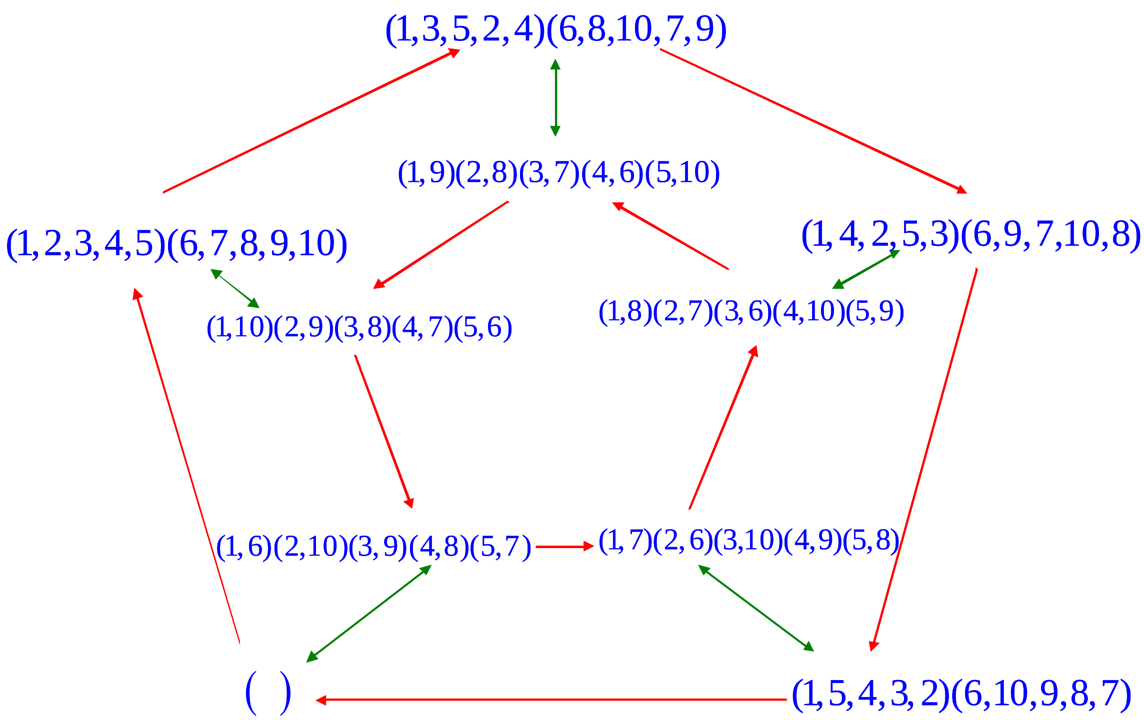
Cycle Graph



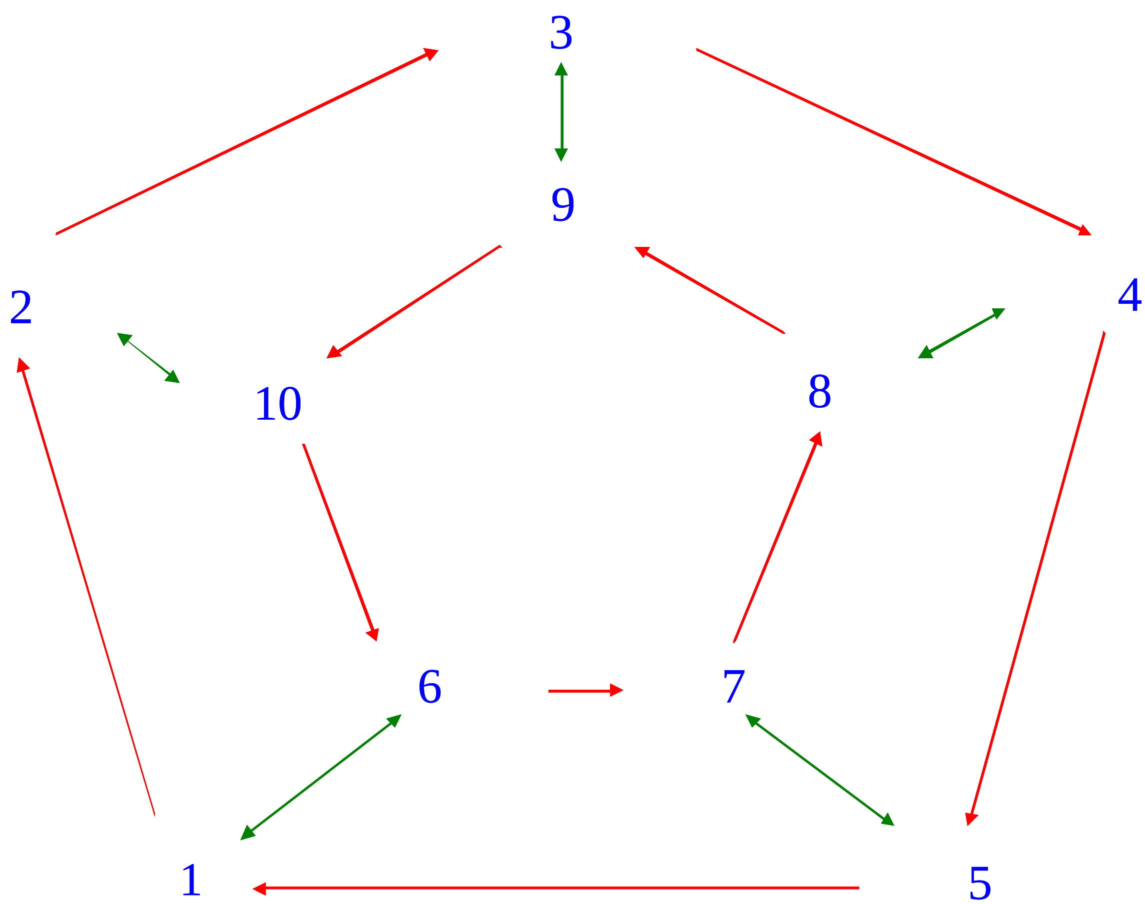
Cayley Diagram



or


(1,3,5,2,4)(6,8,10,7,9)
(1,9)(2,8)(3,7)(4,6)(5,10)
(1,2,3,4,5)(6,7,8,9,10)
(1,4,2,5,3)(6,9,7,10,8)
(1,10)(2,9)(3,8)(4,7)(5,6)
(1,8)(2,7)(3,6)(4,10)(5,9)
(1,6)(2,10)(3,9)(4,8)(5,7)
(1,7)(2,6)(3,10)(4,9)(5,8)
( )
(1,5,4,3,2)(6,10,9,8,7)

Generator Diagram



HOW TO USE GAP (PART 5)

Part 5 of *How to Use GAP* is actually the same as Part 4. We are just repeating the information for easy reference.

1. *How can I redisplay the previous command in order to edit it?*

   Press down on the control key and then also press p. In other words, “Ctrl p”.

2. *If the program gets in a loop and shows you the prompt “brk>” instead of “gap>”, how can I exit the loop?*

   Press down on the control key and then also press d. In other words, “Ctrl d”.

3. *How can I exit the program?*

   Either click on the “close” box for the window, or type “quit;” and press “Enter.”

4. *How do I find the inverse of a permutation?*

   ```
   gap> a:=(1,2,3,4);
   (1,2,3,4)
   gap> a^-1;
   (1,4,3,2)
   ```

5. *How can I multiply permutations and raise permutations to powers?*

```
gap> (1,2)*(1,2,3);
(1,3)

gap> (1,2,3)^2;
(1,3,2)

gap> (1,2,3)^-1;
(1,3,2)

gap> (1,2,3)^-2;
(1,2,3)

gap> a:=(1,2,3);
(1,2,3)

gap> b:=(1,2);
(1,2)

gap> a*b;
(2,3)

gap> a^2;
(1,3,2)

gap> a^-2;
(1,2,3)

gap> a^3;
()
```

```
gap> a^-3;
()

gap> (a*b)^2;
()

gap> (a*b)^3;
(2,3)
```

6. *How can I create a group from permutations, find the size of the group, and find the elements in the group?*

```
gap> a:=(1,2);
(1,2)

gap> b:=(1,2,3);
(1,2,3)

gap> g1:=Group(a,b);
Group([ (1,2), (1,2,3) ])

gap> Size(g1);
6

gap> Elements(g1);
[ (), (2,3), (1,2), (1,2,3), (1,3,2), (1,3) ]

gap> g2:=Group([(1,2),(1,2,3)]);
Group([ (1,2), (1,2,3) ])
```

```
gap> g3:=Group((1,2),(2,3,4));
Group([ (1,2), (2,3,4) ])
```

7. *How can I create a cyclic group of order 3?*

```
gap> a:=(1,2,3);
(1,2,3)

gap> g1:=Group(a);
Group([ (1,2,3) ])

gap> Size(g1);
3

gap> Elements(g1);
[ (), (1,2,3), (1,3,2) ]

gap> g2:=Group((1,2,3));
Group([ (1,2,3) ])

gap> g3:=CyclicGroup(IsPermGroup,3);
Group([ (1,2,3) ])
```

8. *How can I create a multiplication table for the cyclic group of order 3 that I just created?*

```
gap> ShowMultiplicationTable(g1);

*       | ()       (1,2,3)  (1,3,2)
--------+---------------------------------
()      | ()       (1,2,3)  (1,3,2)
(1,2,3) | (1,2,3)  (1,3,2)  ()
(1,3,2) | (1,3,2)  ()       1,2,3)
```

9. *How do I determine if a group is abelian?*

```
gap> g1:=Group((1,2,3));
Group([ (1,2,3) ])

gap> IsAbelian(g1);
true

gap> g2:=Group((1,2),(1,2,3));
Group([ (1,2), (1,2,3) ])

gap> IsAbelian(g2);
false
```

10. *What do I type in order to get help for a command like "Elements?"*

```
gap> ?Elements
```

11. *How do I find all subgroups of a group?*

```
gap> a:=(1,2,3);
(1,2,3)
```

```
gap> b:=(2,3);
(2,3)

gap> g:=Group(a,b);
Group([ (1,2,3), (2,3) ])

gap> Size(g);
6

gap> Elements(g);
[ (), (2,3), (1,2), (1,2,3), (1,3,2), (1,3) ]

gap> h:=AllSubgroups(g);
[ Group(()), Group([ (2,3) ]), Group([ (1,2) ]), Group([ (1,3) ]),
Group([ (1,2,3) ]), Group([ (1,2,3), (2,3) ]) ]

gap> List(h,i->Elements(i));
[ [ () ], [ (), (2,3) ], [ (), (1,2) ], [ (), (1,3) ], [ (), (1,2,3),
(1,3,2) ], [ (), (2,3), (1,2), (1,2,3), (1,3,2), (1,3) ] ]

gap> Elements(h[1]);
[ () ]

gap> Elements(h[2]);
[ (), (2,3) ]

gap> Elements(h[3]);
[ (), (1,2) ]

gap> Elements(h[4]);
[ (), (1,3) ]

gap> Elements(h[5]);
[ (), (1,2,3), (1,3,2) ]

gap> Elements(h[6]);
[ (), (2,3), (1,2), (1,2,3), (1,3,2), (1,3) ]
```

12. *How do I find the subgroup generated by particular permutations?*

```
gap> g:=Group((1,2),(1,2,3));
Group([ (1,2), (1,2,3) ])

gap> Elements(g);
[ (), (2,3), (1,2), (1,2,3), (1,3,2), (1,3) ]

gap> h:=Subgroup(g,[(1,2)]);
Group([ (1,2) ])

gap> Elements(h);
[ (), (1,2) ]
```

13. *How do I determine if a subgroup is normal?*

```
gap> g:=Group((1,2),(1,2,3));
Group([ (1,2), (1,2,3) ])

gap> h1:=Group((1,2));
Group([ (1,2) ])
```

```
gap> IsNormal(g,h1);

gap> h2:=Group((1,2,3));
Group([ (1,2,3) ])

gap> IsNormal(g,h2);
true
```

14. *How do I find all normal subgroups of a group?*

```
gap> g:=Group((1,2),(1,2,3));
Group([ (1,2), (1,2,3) ])

gap> Elements(g);
[ (), (2,3), (1,2), (1,2,3), (1,3,2), (1,3) ]

gap> n:=NormalSubgroups(g);
[ Group([ (1,2), (1,2,3) ]), Group([ (1,3,2) ]), Group(()) ]

gap> Elements(n[1]);
[ (), (2,3), (1,2), (1,2,3), (1,3,2), (1,3) ]

gap> Elements(n[2]);
[ (), (1,2,3), (1,3,2) ]

gap> Elements(n[3]);
[ () ]
```

15. *How do I determine if a group is simple?*

```
gap> g:=Group((1,2),(1,2,3));
Group([ (1,2), (1,2,3) ])

gap> Elements(g);
[ (), (2,3), (1,2), (1,2,3), (1,3,2), (1,3) ]

gap> IsSimple(g);
false

gap> h:=Group((1,2));
Group([ (1,2) ])

gap> Elements(h);
[ (), (1,2) ]

gap> IsSimple(h);
true
```

16. *How do I find the right cosets of a subset H of G?*

```
gap> g:=Group([(1,2,3),(1,2)]);
Group([ (1,2,3), (1,2) ])

gap> Elements(g);
[ (), (2,3), (1,2), (1,2,3), (1,3,2), (1,3) ]

gap> h:=Subgroup(g,[(1,2)]);
Group([ (1,2) ])

gap> Elements(h);
[ (), (1,2) ]

gap> c:=RightCosets(g,h);
[ RightCoset(Group( [ (1,2) ] ),()), RightCoset(Group( [ (1,2) ] ),(1,3,2)),
RightCoset(Group( [ (1,2) ] ),(1,2,3)) ]

gap> List(c,i->Elements(i));
[ [ (), (1,2) ], [ (2,3), (1,3,2) ], [ (1,2,3), (1,3) ] ]
gap> Elements(c[1]);
[ (), (1,2) ]

gap> Elements(c[2]);
[ (2,3), (1,3,2) ]


gap> Elements(c[3]);
[ (1,2,3), (1,3) ]

gap> rc:=RightCoset(h,(1,2,3));
RightCoset(Group([ (1,2) ]),(1,2,3))

gap> Elements(rc);
[ (1,2,3), (1,3) ]

gap> rc:=h*(1,2,3);
RightCoset(Group([ (1,2) ]),(1,2,3))

gap> Elements(rc);
[ (1,2,3), (1,3) ]
```

17. *How can I create a quotient (factor) group?*

```
gap> g:=Group([(1,2,3),(1,2)]);
Group([ (1,2,3), (1,2) ])

gap> Elements(g);
[ (), (2,3), (1,2), (1,2,3), (1,3,2), (1,3) ]

gap> n:=Group((1,2,3));
Group([ (1,2,3) ])

gap> Elements(n);
[ (), (1,2,3), (1,3,2) ]

gap> IsNormal(g,n);
true

gap> c:=RightCosets(g,n);
[ RightCoset(Group([ (1,2,3) ]),()), RightCoset(Group([ (1,2,3) ]),(2,3)) ]
```

```
gap> Elements(c[1]);
[ (), (1,2,3), (1,3,2) ]

gap> Elements(c[2]);
[ (2,3), (1,2), (1,3) ]

gap> f:=FactorGroup(g,n);
Group([ f1 ])

gap> Elements(f);
[ <identity> of ..., f1 ]

gap> ShowMultiplicationTable(f);
*                  | <identity> of ... f1
-------------------+----------------------------------
<identity> of ... | <identity> of ... f1
f1                 | f1                 <identity> of ...
```

18. *How do I find the center of a group?*

```
gap> a:=(1,2,3);
(1,2,3)

gap> b:=(2,3);
(2,3)

gap> g:=Group(a,b);
Group([ (1,2,3), (2,3) ])

gap> Center(g);
Group(())

gap> c:=Center(g);
Group(())

gap> Elements(c);
[ () ]


gap> a:=(1,2,3,4);
(1,2,3,4)

gap> b:=(1,3);
(1,3)

gap> g:=Group(a,b);
Group([ (1,2,3,4), (1,3) ])

gap> c:=Center(g);
Group([ (1,3)(2,4) ])

gap> Elements(c);
[ (), (1,3)(2,4) ]
```

19. *How do I find the commutator (derived) subgroup of a group?*

```
gap> a:=(1,2,3);
(1,2,3)
```

```
gap> b:=(2,3);
(2,3)

gap> g:=Group(a,b);
Group([ (1,2,3), (2,3) ])

gap> d:=DerivedSubgroup(g);
Group([ (1,3,2) ])

gap> Elements(d);
[ (), (1,2,3), (1,3,2) ]


gap> a:=(1,2,3,4);
(1,2,3,4)

gap> b:=(1,3);
(1,3)

gap> g:=Group(a,b);
Group([ (1,2,3,4), (1,3) ])

gap> d:=DerivedSubgroup(g);
Group([ (1,3)(2,4) ])

gap> Elements(d);
[ (), (1,3)(2,4) ]
```

20. *How do I find all Sylow p-subgroups for a given group?*

```
gap> a:=(1,2,3);
(1,2,3)

gap> b:=(2,3);
(2,3)

gap> g:=Group(a,b);
Group([ (1,2,3), (2,3) ])

gap> Size(g);
6

gap> FactorsInt(6);
[ 2, 3 ]

gap> sylow2:=SylowSubgroup(g,2);
Group([ (2,3) ])

gap> IsNormal(g,sylow2);
false

gap> c:=ConjugateSubgroups(g,sylow2);
[ Group([ (2,3) ]), Group([ (1,3) ]), Group([ (1,2) ]) ]

gap> Elements(c[1]);
[ (), (2,3) ]

gap> Elements(c[2]);
[ (), (1,3) ]

gap> Elements(c[3]);
[ (), (1,2) ]

gap> sylow3:=SylowSubgroup(g,3);
Group([ (1,2,3) ])
```

```
gap> IsNormal(g,sylow3);
true

gap> Elements(sylow3);
[ (), (1,2,3), (1,3,2) ]
```

21. *How can I create the Rubik's cube group using GAP?*

First you need to save the following permutations as a pure text file with the name rubik.txt to your C-drive before you can import it into GAP.

```
r:=(25,27,32,30)(26,29,31,28)(3,38,43,19)(5,36,45,21)(8,33,48,24);
l:=(9,11,16,14)(10,13,15,12)(1,17,41,40)(4,20,44,37)(6,22,46,35);
u:=(1,3,8,6)(2,5,7,4)(9,33,25,17)(10,34,26,18)(11,35,27,19);
d:=(41,43,48,46)(42,45,47,44)(14,22,30,38)(15,23,31,39)(16,24,32,40);
f:=(17,19,24,22)(18,21,23,20)(6,25,43,16)(7,28,42,13)(8,30,41,11);
b:=(33,35,40,38)(34,37,39,36)(3,9,46,32)(2,12,47,29)(1,14,48,27);
```

And now you can read the file into GAP and begin exploring.

```
gap> Read("C:/rubik.txt");

gap> rubik:=Group(r,l,u,d,f,b);
<permutation group with 6 generators>

gap> Size(rubik);
43252003274489856000
```

22. *How can I find the center of the Rubik's cube group?*

```
gap> c:=Center(rubik);
Group([ (2,34)(4,10)(5,26)(7,18)(12,37)(13,20)(15,44)(21,28)(23,42)(29,36)(31,4
5)(39,47) ])

gap> Size(c);
2

gap> Elements(c);
[ (), (2,34)(4,10)(5,26)(7,18)(12,37)(13,20)(15,44)(21,28)(23,42)(29,36)(31,45)
(39,47) ]
```

23. *How can I find the commutator (derived) subgroup of the Rubik's cube group?*

```
gap> d:=DerivedSubgroup(rubik);
<permutation group with 5 generators>

gap> Size(d);
21626001637244928000

gap> IsNormal(rubik,d);
true
```

24. *How can I find the quotient (factor) group of the Rubik's cube group by its commutator (derived) subgroup?*

```
gap> d:=DerivedSubgroup(rubik);
<permutation group of size 21626001637244928000 with 5 generators>

gap> f:=FactorGroup(rubik,d);
Group([ f1 ])

gap> Size(f);
2
```

25. *How can I find some Sylow p-subgroups of the Rubik's cube group?*

```
gap> Read("C:/rubik.txt");

gap> rubik:=Group(r,l,u,d,f,b);
<permutation group with 6 generators>

gap> Size(rubik);
43252003274489856000

gap> FactorsInt(43252003274489856000);
[ 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2,
2, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 5, 5, 5, 7, 7, 11 ]

gap> sylow2:=SylowSubgroup(rubik,2);
<permutation group of size 134217728 with 27 generators>

gap> sylow3:=SylowSubgroup(rubik,3);
<permutation group of size 4782969 with 14 generators>

gap> sylow5:=SylowSubgroup(rubik,5);
<permutation group of size 125 with 3 generators>

gap> sylow7:=SylowSubgroup(rubik,7);
<permutation group of size 49 with 2 generators>

gap> sylow11:=SylowSubgroup(rubik,11);
Group([ (4,36,31,39,42,12,5,21,15,13,7)(10,29,45,47,23,37,26,28,44,20,18) ])
```

```
gap> Elements(sylow11);
[ (), (4,5,36,21,31,15,39,13,42,7,12)(10,26,29,28,45,44,47,20,23,18,37),
(4,7,13,15,21,5,12,42,39,31,36)(10,18,20,44,28,26,37,23,47,45,29),
(4,12,7,42,13,39,15,31,21,36,5)(10,37,18,23,20,47,44,45,28,29,26),
(4,13,21,12,39,36,7,15,5,42,31)(10,20,28,37,47,29,18,44,26,23,45),
(4,15,12,31,7,21,42,36,13,5,39)(10,44,37,45,18,28,23,29,20,26,47),
(4,21,39,7,5,31,13,12,36,15,42)(10,28,47,18,26,45,20,37,29,44,23),
(4,31,42,5,15,7,36,39,12,21,13)(10,45,23,26,44,18,29,47,37,28,20),
(4,36,31,39,42,12,5,21,15,13,7)(10,29,45,47,23,37,26,28,44,20,18),
(4,39,5,13,36,42,21,7,31,12,15)(10,47,26,20,29,23,28,18,45,37,44),
(4,42,15,36,12,13,31,5,7,39,21)(10,23,44,29,37,20,45,26,18,47,28) ]

gap> IsNormal(rubik,sylow2);
false

gap> IsNormal(rubik,sylow3);
false


gap> IsNormal(rubik,sylow5);
false

gap> IsNormal(rubik,sylow7);
false

gap> IsNormal(rubik,sylow11);
false
```

NOTE: All of the *Sylow p-subgroups* found above have *conjugates*, but the sheer size of the *Rubik's cube group* makes it too difficult to pursue them on a typical desktop computer.

26. *How do I determine if a group is cyclic?*

```
gap> a:=(1,2,3)*(4,5,6,7);
(1,2,3)(4,5,6,7)

gap> g:=Group(a);
Group([ (1,2,3)(4,5,6,7) ])

gap> Size(g);
12

gap> IsCyclic(g);
true
```

27. *How do I create a dihedral group with 2n elements for an n-sided regular polygon?*

```
gap> d4:=DihedralGroup(IsPermGroup,8);
Group([ (1,2,3,4), (2,4) ])
```

```
gap> Elements(d4);
[ (), (2,4), (1,2)(3,4), (1,2,3,4), (1,3), (1,3)(2,4), (1,4,3,2), (1,4)(2,3) ]
```

28. *How can I express the elements of a dihedral group as rotations and flips rather than as permutations?*

```
gap> d3:=DihedralGroup(6);
<pc group of size 6 with 2 generators>

gap> Elements(d3);
[ <identity> of ..., f1, f2, f1*f2, f2^2, f1*f2^2 ]

gap> ShowMultiplicationTable(d3);
*                 | <identity> of ... f1                f2                f1*f2             f2^2              f1*f2^2
------------------+-----------------------------------------------------------------------------------------------------\
---------
<identity> of ... | <identity> of ... f1                f2                f1*f2             f2^2              f1*f2^2
f1                | f1                <identity> of ... f1*f2             f2                f1*f2^2           f2^2
f2                | f2                f1*f2^2           f2^2              f1                <identity> of ... f1*f2
f1*f2             | f1*f2             f2^2              f1*f2^2           <identity> of ... f1                f2
f2^2              | f2^2              f1*f2             <identity> of ... f1*f2^2           f2                f1
f1*f2^2           | f1*f2^2           f2                f1                f2^2              f1*f2
<identity> of ...
```

29. *How do I create a symmetric group of degree n with n! elements?*

```
gap> s4:=SymmetricGroup(4);
Sym( [ 1 .. 4 ] )

gap> Size(s4);
24

gap> Elements(s4);
[ (), (3,4), (2,3), (2,3,4), (2,4,3), (2,4), (1,2), (1,2)(3,4), (1,2,3),
(1,2,3,4), (1,2,4,3), (1,2,4), (1,3,2),
  (1,3,4,2), (1,3), (1,3,4), (1,3)(2,4), (1,3,2,4), (1,4,3,2), (1,4,2), (1,4,3),
(1,4), (1,4,2,3), (1,4)(2,3) ]
```

30. *How do I create an alternating group of degree n with* $\frac{n!}{2}$ *elements?*

```
gap> a4:=AlternatingGroup(4);
Alt( [ 1 .. 4 ] )

gap> Size(a4);
12
gap> Elements(a4);
[ (), (2,3,4), (2,4,3), (1,2)(3,4), (1,2,3), (1,2,4), (1,3,2), (1,3,4),
(1,3)(2,4), (1,4,2), (1,4,3), (1,4)(2,3) ]
```

31. *How do I create a direct product of two or more groups?*

```
gap> g1:=Group((1,2,3));
Group([ (1,2,3) ])
```

```
gap> g2:=Group((4,5));
Group([ (4,5) ])

gap> dp:=DirectProduct(g1,g2);
Group([ (1,2,3), (4,5) ])

gap> Size(dp);
6
gap> Elements(dp);
[ (), (4,5), (1,2,3), (1,2,3)(4,5), (1,3,2), (1,3,2)(4,5) ]

gap> ShowMultiplicationTable(dp);
*            | ()           (4,5)        (1,2,3)      (1,2,3)(4,5) (1,3,2)
(1,3,2)(4,5)
-------------+----------------------------------------------------------------------
---
()           | ()           (4,5)        (1,2,3)      (1,2,3)(4,5) (1,3,2)
(1,3,2)(4,5)
(4,5)        | (4,5)        ()           (1,2,3)(4,5) (1,2,3)      (1,3,2)(4,5) (1,3,2)
(1,2,3)      | (1,2,3)      (1,2,3)(4,5) (1,3,2)      (1,3,2)(4,5) ()           (4,5)
(1,2,3)(4,5) | (1,2,3)(4,5) (1,2,3)      (1,3,2)(4,5) (1,3,2)      (4,5)        ()
(1,3,2)      | (1,3,2)      (1,3,2)(4,5) ()           (4,5)        (1,2,3)
(1,2,3)(4,5)
(1,3,2)(4,5) | (1,3,2)(4,5) (1,3,2)      (4,5)        ()           (1,2,3)(4,5) (1,2,3)
```

32. *How can I create the Quaternion group?*

```
gap> a:=(1,2,5,6)*(3,8,7,4);
(1,2,5,6)(3,8,7,4)

gap> b:=(1,4,5,8)*(2,7,6,3);
(1,4,5,8)(2,7,6,3)

gap> q:=Group(a,b);
Group([ (1,2,5,6)(3,8,7,4), (1,4,5,8)(2,7,6,3) ])

gap> Size(q);
8

gap> IsAbelian(q);
false

gap> Elements(q);
[ (), (1,2,5,6)(3,8,7,4), (1,3,5,7)(2,4,6,8), (1,4,5,8)(2,7,6,3),
(1,5)(2,6)(3,7)(4,8), (1,6,5,2)(3,4,7,8),
  (1,7,5,3)(2,8,6,4), (1,8,5,4)(2,3,6,7) ]

gap> q:=QuaternionGroup(IsPermGroup,8);
Group([ (1,5,3,7)(2,8,4,6), (1,2,3,4)(5,6,7,8) ])

gap> Size(q);
8


gap> IsAbelian(q);
false

gap> Elements(q);
[ (), (1,2,3,4)(5,6,7,8), (1,3)(2,4)(5,7)(6,8), (1,4,3,2)(5,8,7,6),
(1,5,3,7)(2,8,4,6), (1,6,3,8)(2,5,4,7),
  (1,7,3,5)(2,6,4,8), (1,8,3,6)(2,7,4,5) ]
```

33. *How can I find a set of independent generators for a group?*

```
gap> c6:=CyclicGroup(IsPermGroup,6);
Group([ (1,2,3,4,5,6) ])

gap> Size(c6);
6

gap> GeneratorsOfGroup(c6);
[ (1,2,3,4,5,6) ]



gap> d4:=DihedralGroup(IsPermGroup,8);
Group([ (1,2,3,4), (2,4) ])

gap> Size(d4);
8

gap> GeneratorsOfGroup(d4);
[ (1,2,3,4), (2,4) ]



gap> s5:=SymmetricGroup(5);
Sym( [ 1 .. 5 ] )


gap> Size(s5);
120

gap> GeneratorsOfGroup(s5);
[ (1,2,3,4,5), (1,2) ]


gap> a5:=AlternatingGroup(5);
Alt( [ 1 .. 5 ] )

gap> Size(a5);
60

gap> GeneratorsOfGroup(a5);
[ (1,2,3,4,5), (3,4,5) ]



gap> q:=QuaternionGroup(IsPermGroup,8);
Group([ (1,5,3,7)(2,8,4,6), (1,2,3,4)(5,6,7,8) ])

gap> Size(q);
8

gap> GeneratorsOfGroup(q);
[ (1,5,3,7)(2,8,4,6), (1,2,3,4)(5,6,7,8) ]
```

34. *How do I find the conjugate of a permutation in the form* $a^b = b^{-1}ab$ *?*

```
gap> a:=(1,2,3,4,5);
(1,2,3,4,5)

gap> b:=(2,4,5);
(2,4,5)
```

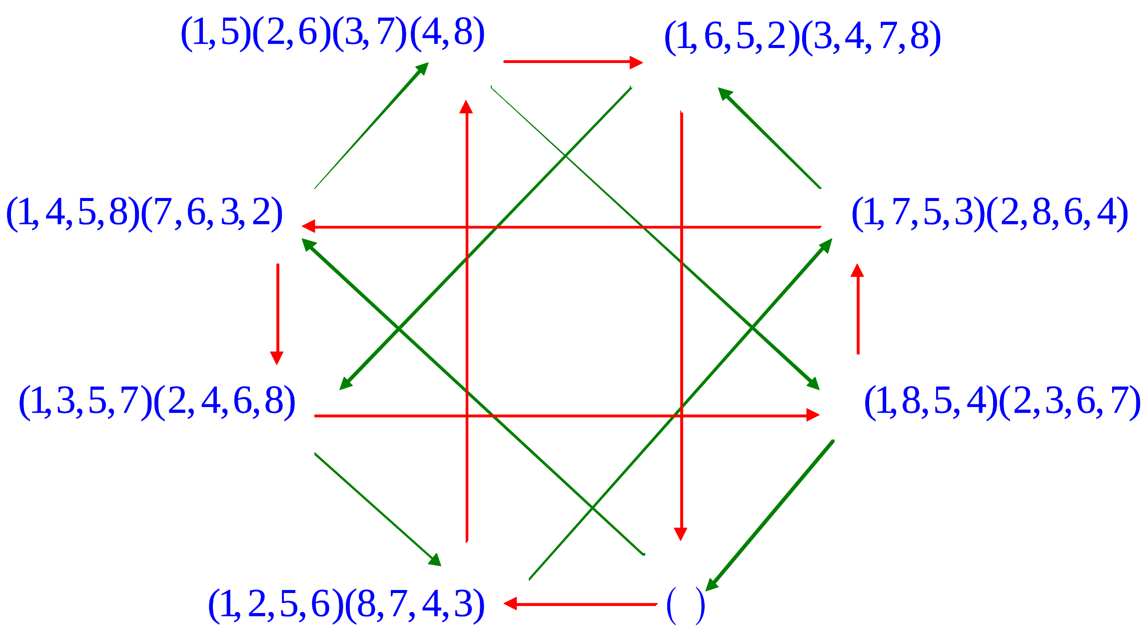
```
gap> a^b;
(1,4,3,5,2)

gap> b^-1*a*b;
(1,4,3,5,2)
```

35. *How do I divide up a group into classes of elements that are conjugate to one another? (Note that "conjugacy" is an equivalence relation on our group G. That means that G can be separated into nonintersecting subsets that contain only elements that are conjugate to one another.)*

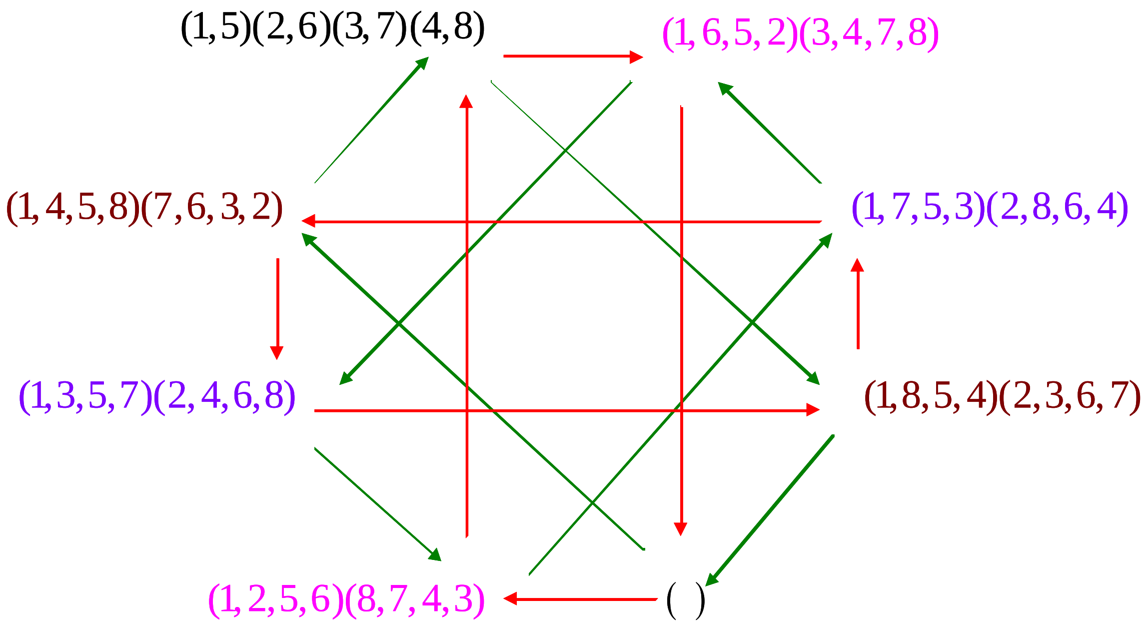
```
gap> d3:=DihedralGroup(IsPermGroup,6);
Group([ (1,2,3), (2,3) ])

gap> Size(d3);
6

gap> Elements(d3);
[ (), (2,3), (1,2), (1,2,3), (1,3,2), (1,3) ]

gap> cc:=ConjugacyClasses(d3);
[ ()^G, (2,3)^G, (1,2,3)^G ]

gap> Elements(cc[1]);
[ () ]

gap> Elements(cc[2]);
[ (2,3), (1,2), (1,3) ]

gap> Elements(cc[3]);
[ (1,2,3), (1,3,2) ]
```

HOW TO FIND THE QUOTIENT OF A CAYLEY DIAGRAM

We're now going to give an illustration of how to find the *Cayley diagram* of a *quotient group*, and for this example we will use the *Quanternion Group* since all of its *subgroups* are *normal subgroups*. Thus, below is the *Cayley diagram* for the *Quanternion Group*, and the *subgroup* we are going to factor our is $\begin{Bmatrix} (\ ) \\ (1,5)(2,6)(3,7)(4,8) \end{Bmatrix}$.



And now, for convenience, we'll color-code the various *right cosets* that are to be found in our corresponding *quotient group*.



The four elements in our *quotient group* can be listed as follows.

$\begin{Bmatrix} (\ ) \\ (1,5)(2,6)(3,7)(4,8) \end{Bmatrix}$, $\begin{Bmatrix} (1,2,5,6)(8,7,4,3) \\ (1,6,5,2)(3,4,7,8) \end{Bmatrix}$, $\begin{Bmatrix} (1,7,5,3)(2,8,6,4) \\ (1,3,5,7)(2,4,6,8) \end{Bmatrix}$, and

$\begin{Bmatrix} (1,4,5,8)(7,6,3,2) \\ (1,8,5,4)(2,3,6,7) \end{Bmatrix}$

We can also write each of those as a *right coset* of the *normal subgroup* that we are factoring out.

$$\begin{Bmatrix} (\ ) \\ (1,5)(2,6)(3,7)(4,8) \end{Bmatrix}$$

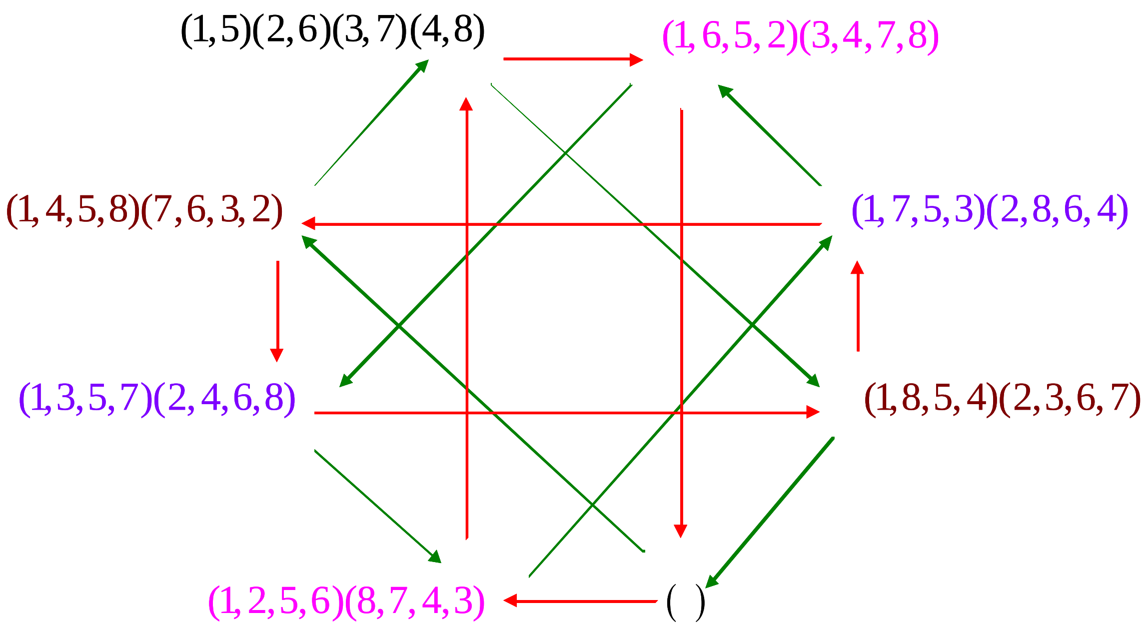
$$\begin{Bmatrix} (1,2,5,6)(8,7,4,3) \\ (1,6,5,2)(3,4,7,8) \end{Bmatrix} = \begin{Bmatrix} (\ ) \\ (1,5)(2,6)(3,7)(4,8) \end{Bmatrix}(1,2,5,6)(8,7,4,3)$$

$$\begin{Bmatrix} (1,4,5,8)(7,6,3,2) \\ (1,8,5,4)(2,3,6,7) \end{Bmatrix} = \begin{Bmatrix} (\ ) \\ (1,5)(2,6)(3,7)(4,8) \end{Bmatrix}(1,4,5,8)(7,6,3,2)$$

$$\begin{Bmatrix} (1,7,5,3)(2,8,6,4) \\ (1,3,5,7)(2,4,6,8) \end{Bmatrix} = \begin{Bmatrix} (\ ) \\ (1,5)(2,6)(3,7)(4,8) \end{Bmatrix}(1,7,5,3)(2,8,6,4)$$

$$= \begin{Bmatrix} (\ ) \\ (1,5)(2,6)(3,7)(4,8) \end{Bmatrix}(1,2,5,6)(8,7,4,3)(1,4,5,8)(7,6,3,2)$$

Using our Cayley diagram where the *cosets* are color-coded (repeated below), we can find the order of each element of our *quotient group* by following the appropriate arrows leading away from each particular *coset*.



Thus, if we apply the red arrows twice to each element of the *coset* $\left\{\begin{matrix}(\ )\\(1,5)(2,6)(3,7)(4,8)\end{matrix}\right\}$, then we arrive back at elements in the *coset* $\left\{\begin{matrix}(\ )\\(1,5)(2,6)(3,7)(4,8)\end{matrix}\right\}$. Hence, $\left\{\begin{matrix}(\ )\\(1,5)(2,6)(3,7)(4,8)\end{matrix}\right\}(1,2,5,6)(8,7,4,3)$ has *order* 2. Similarly, if we apply the green arrows twice to $\left\{\begin{matrix}(\ )\\(1,5)(2,6)(3,7)(4,8)\end{matrix}\right\}$, then we also arrive back at elements in the *coset* $\left\{\begin{matrix}(\ )\\(1,5)(2,6)(3,7)(4,8)\end{matrix}\right\}$, and so $\left\{\begin{matrix}(\ )\\(1,5)(2,6)(3,7)(4,8)\end{matrix}\right\}(1,4,5,8)(7,6,3,2)$ also has *order* 2. Also, if we apply the red arrow followed by the green arrow repeatedly to the *coset* we have factored out, then we again show that we have an element of *order* 2..

$$\begin{Bmatrix} (\ ) \\ (1,5)(2,6)(3,7)(4,8) \end{Bmatrix}$$

$$\begin{Bmatrix} (\ ) \\ (1,5)(2,6)(3,7)(4,8) \end{Bmatrix}(1,2,5,6)(3,8,7,4)(1,4,5,8)(7,6,3,2)$$

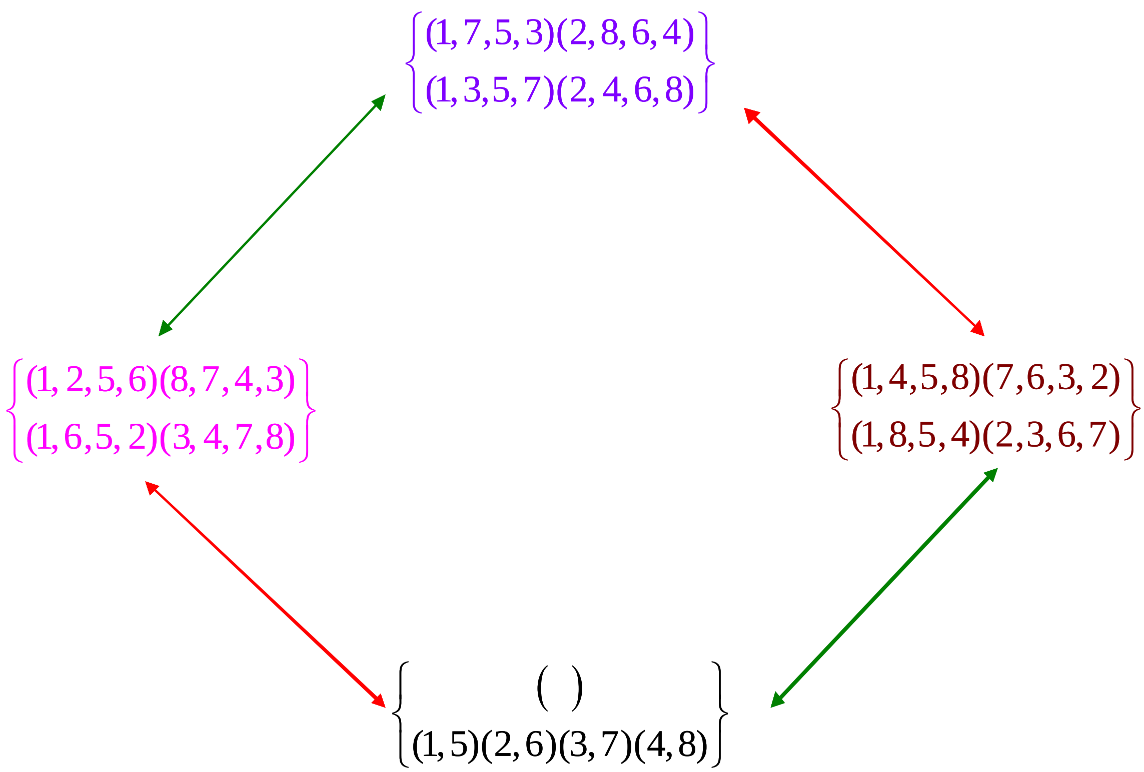
$$=\begin{Bmatrix} (1,7,5,3)(2,8,6,4) \\ (1,3,5,7)(2,4,6,8) \end{Bmatrix}$$

$$\begin{Bmatrix} (1,7,5,3)(2,8,6,4) \\ (1,3,5,7)(2,4,6,8) \end{Bmatrix}(1,2,5,6)(3,8,7,4)(1,4,5,8)(7,6,3,2)$$

$$=\begin{Bmatrix} (\ ) \\ (1,5)(2,6)(3,7)(4,8) \end{Bmatrix}[(1,2,5,6)(3,8,7,4)(1,4,5,8)(7,6,3,2)]^2$$

$$=\begin{Bmatrix} (1,5)(2,6)(3,7)(4,8) \\ (\ ) \end{Bmatrix}$$

From this we know that our *quotient group* must be *isomorphic* to the *Klein* 4-*group*, and we can easily construct its *Cayley diagram*. And we are done.



Summary (Part 5)

Upon completing this part, you should be able to construct the following items for a *group* once you are given generators for that *group*:

- A *cycle graph*.
- A *Cayley diagram*.
- A *generator diagram.*
- A *canonical generator diagram.*
- The *quotient* of a *Cayley diagram*

The fun continues!

PRACTICE (PART 5)

For each given *group* and set of generators for that *group*, construct the same type of visual analysis presented in Part 5 of this work.

1. *Group*: $C_{11} \cong \mathbb{Z}_{11}$ (*cyclic group*)

   Generators: $(1,2,3,4,5,6,7,8,9,10,11)$

2. *Group*: $C_6 \times C_2 \cong C_3 \times C_2 \times C_2 \cong \mathbb{Z}_3 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \cong \mathbb{Z}_6 \times \mathbb{Z}_2$ (*direct product*)

   Generators: $(1,2,3,4,5,6),(7,8)$

3. *Group*: $D_6 \cong \mathbb{Z}_6 >\!\!\lhd \mathbb{Z}_2 \cong C_6 >\!\!\lhd C_2$ (*dihedral group*)

   Generators: $(1,2,3,4,5,6),(2,6)(3,5)$

4. *Group*: A_4 (*alternating group*)

   Generators: $(1,2,3),(2,3,4)$

Practice (Part 5) - Answers

For each given *group* and set of generators for that *group*, construct the same type of visual analysis presented in part 5 of this work.

1. *Group*: $C_{11} \cong \mathbb{Z}_{11}$ (*cyclic group*)

   Generators: $(1,2,3,4,5,6,7,8,9,10,11)$

THE *CYCLIC GROUP* OF ORDER 11

$$C_{11} \cong \mathbb{Z}_{11}$$

Generators:

$(1,2,3,4,5,6,7,8,9,10,11)$

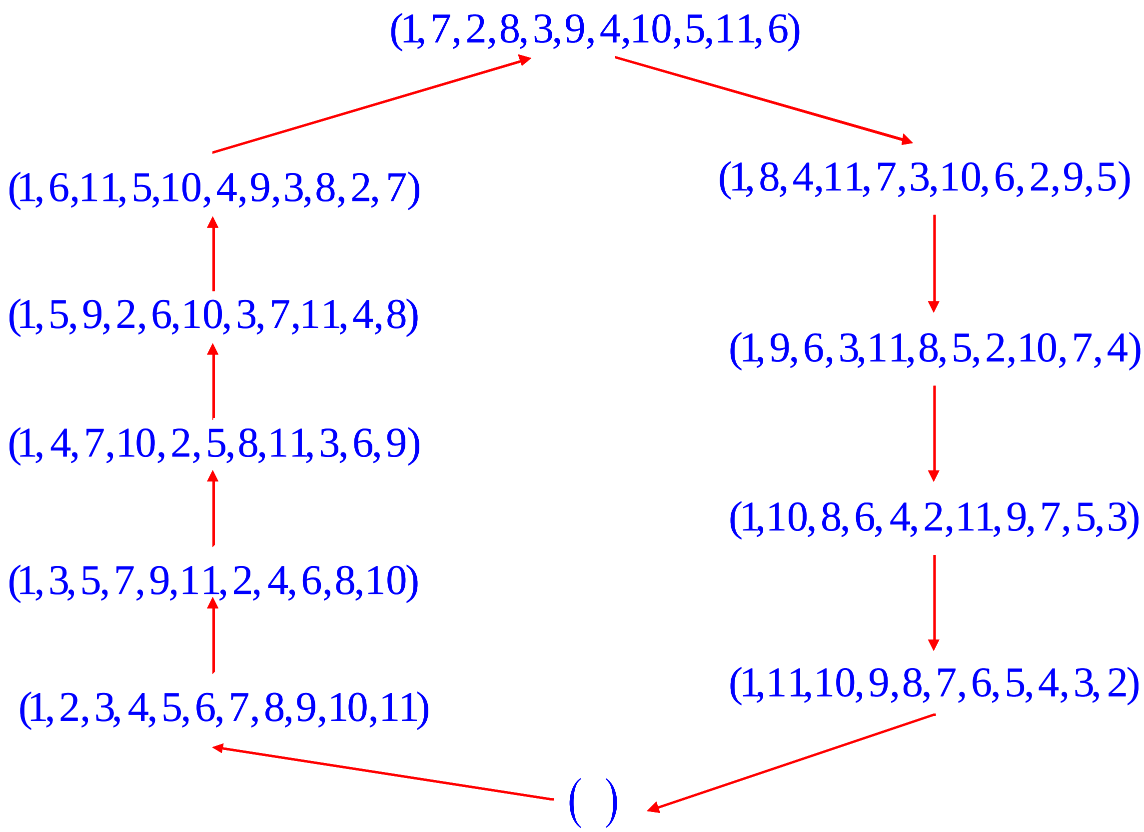
Elements:

$$\left\{\begin{matrix} (\ ) \\ (1,2,3,4,5,6,7,8,9,10,11) \\ (1,3,5,7,9,11,2,4,6,8,10) \\ (1,4,7,10,2,5,8,11,3,6,9) \\ (1,5,9,2,6,10,3,7,11,4,8) \\ (1,6,11,5,10,4,9,3,8,2,7) \\ (1,7,2,8,3,9,4,10,5,11,6) \\ (1,8,4,11,7,3,10,6,2,9,5) \\ (1,9,6,3,11,8,5,2,10,7,4) \\ (1,10,8,6,4,2,11,9,7,5,3) \\ (1,11,10,9,8,7,6,5,4,3,2) \end{matrix}\right\} \cong C_{11}$$

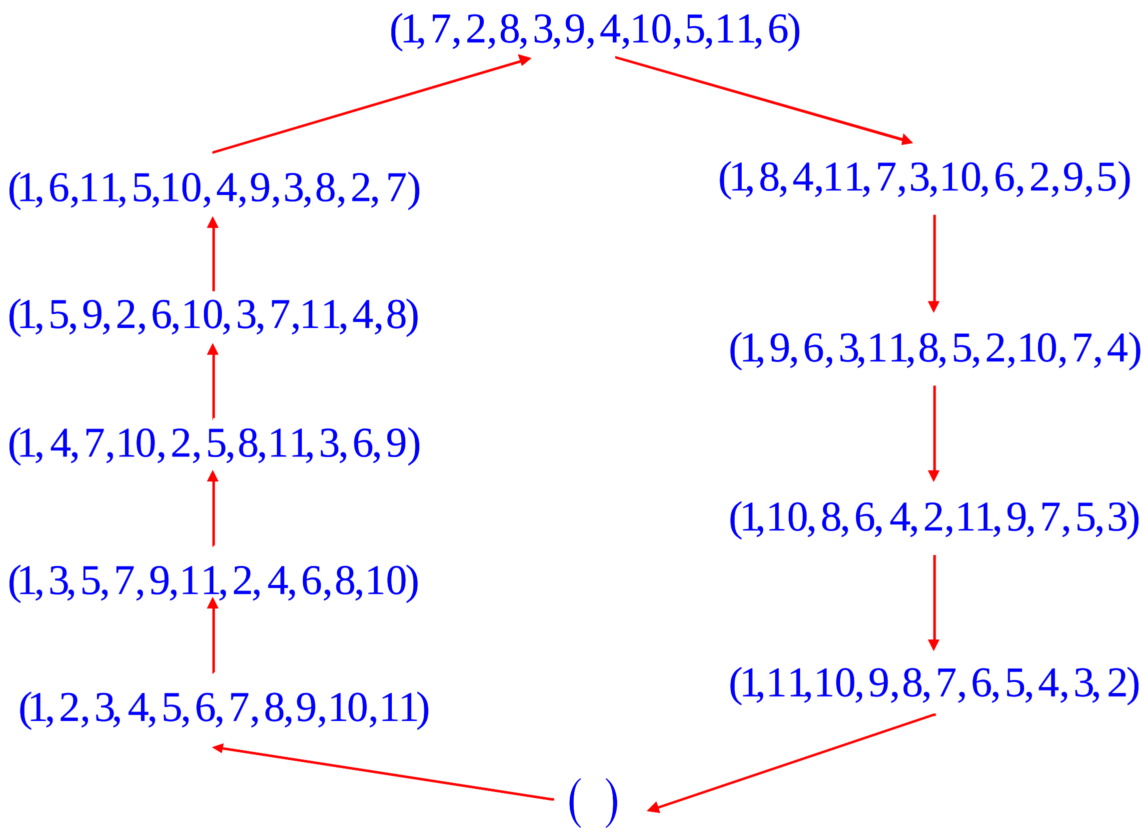
Is *Abelian*?

Yes

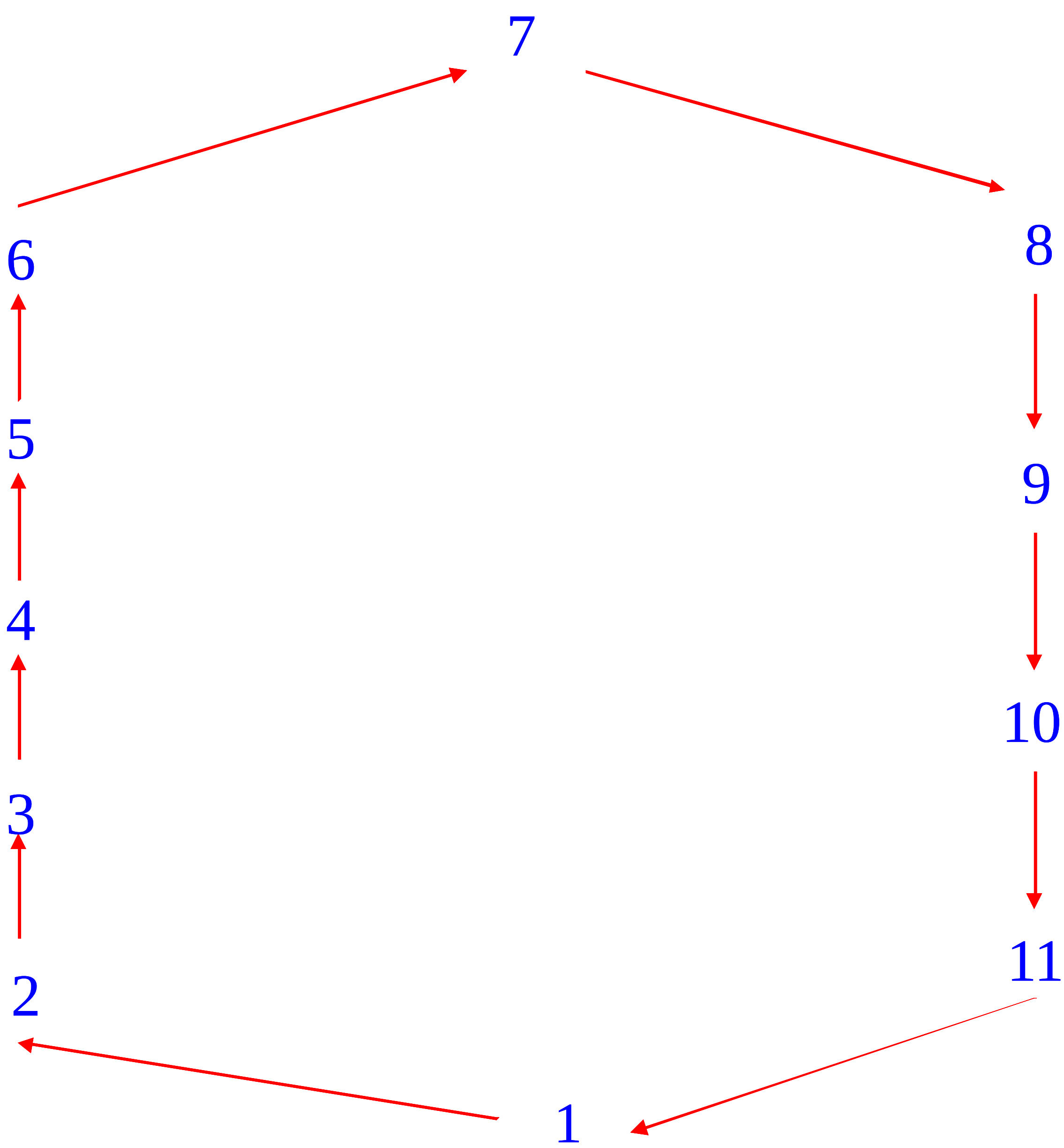
Cycle Graph


(1,7,2,8,3,9,4,10,5,11,6)
(1,6,11,5,10,4,9,3,8,2,7)
(1,8,4,11,7,3,10,6,2,9,5)
(1,5,9,2,6,10,3,7,11,4,8)
(1,9,6,3,11,8,5,2,10,7,4)
(1,4,7,10,2,5,8,11,3,6,9)
(1,10,8,6,4,2,11,9,7,5,3)
(1,3,5,7,9,11,2,4,6,8,10)
(1,11,10,9,8,7,6,5,4,3,2)
(1,2,3,4,5,6,7,8,9,10,11)
( )

Cayley diagram



Generator Diagram



2. *Group*: $C_6 \times C_2 \cong C_3 \times C_2 \times C_2 \cong \mathbb{Z}_3 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \cong \mathbb{Z}_6 \times \mathbb{Z}_2$ (*direct product*)

Generators: $(1,2,3,4,5,6),(7,8)$

THE *DIRECT PRODUCT* $\mathbb{Z}_6 \times \mathbb{Z}_2$

$$C_6 \times C_2 \cong C_3 \times C_2 \times C_2 \cong \mathbb{Z}_3 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \cong \mathbb{Z}_6 \times \mathbb{Z}_2$$

Generators:

$(1,2,3,4,5,6),(7,8)$

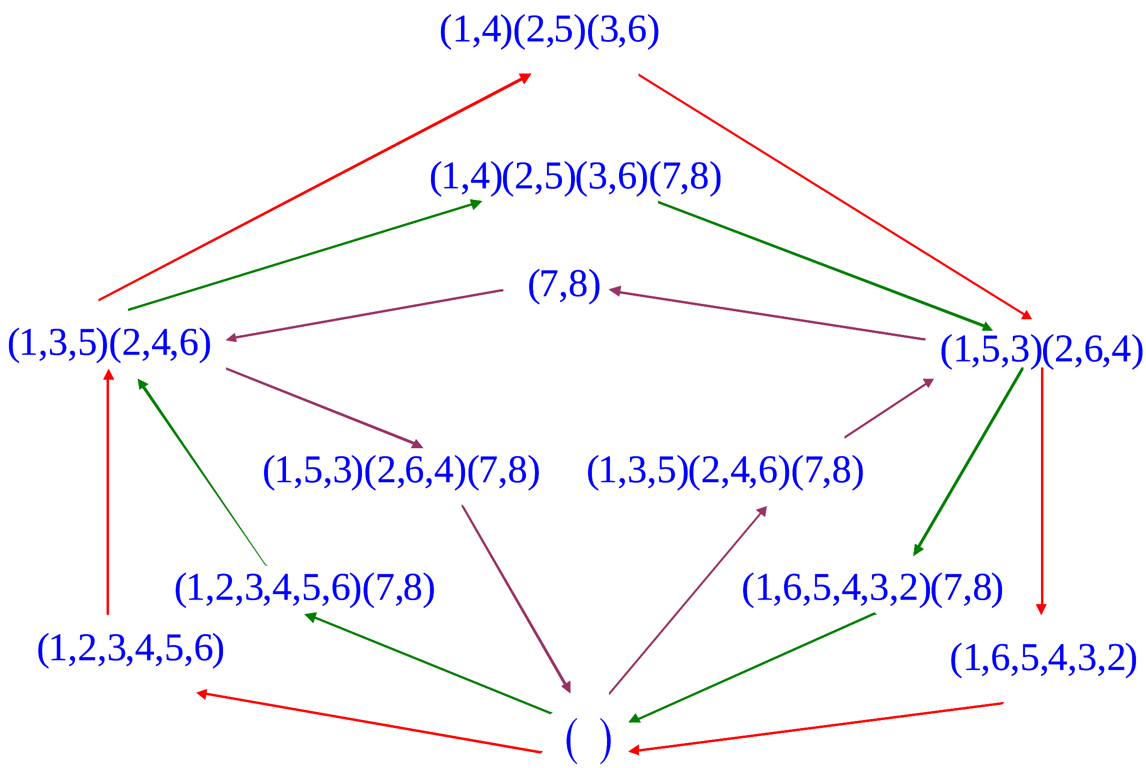
Elements:

$$\left\{ \begin{array}{c} (\ ) \\ (7,8) \\ (1,2,3,4,5,6) \\ (1,2,3,4,5,6)(7,8) \\ (1,3,5)(2,4,6) \\ (1,3,5)(2,4,6)(7,8) \\ (1,4)(2,5)(3,6) \\ (1,4)(2,5)(3,6)(7,8) \\ (1,5,3)(2,6,4) \\ (1,5,3)(2,6,4)(7,8) \\ (1,6,5,4,3,2) \\ (1,6,5,4,3,2)(7,8) \end{array} \right\} \cong C_6 \times C_2$$

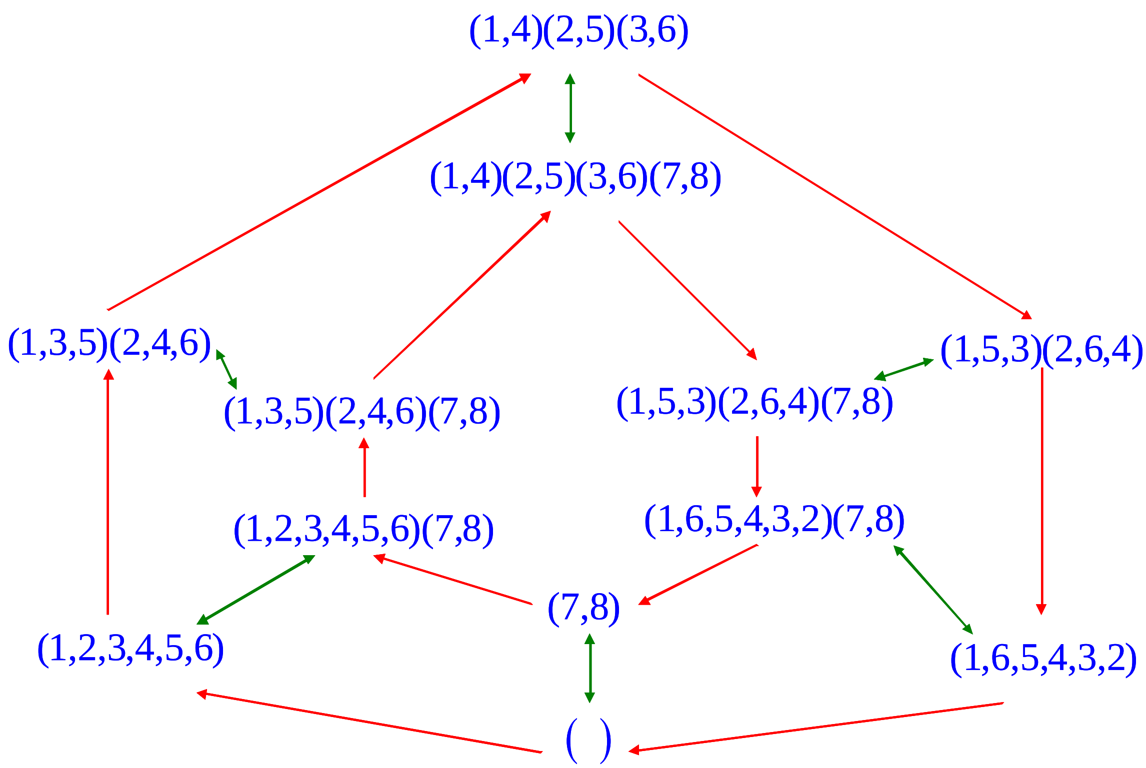
Is *Abelian*?

Yes

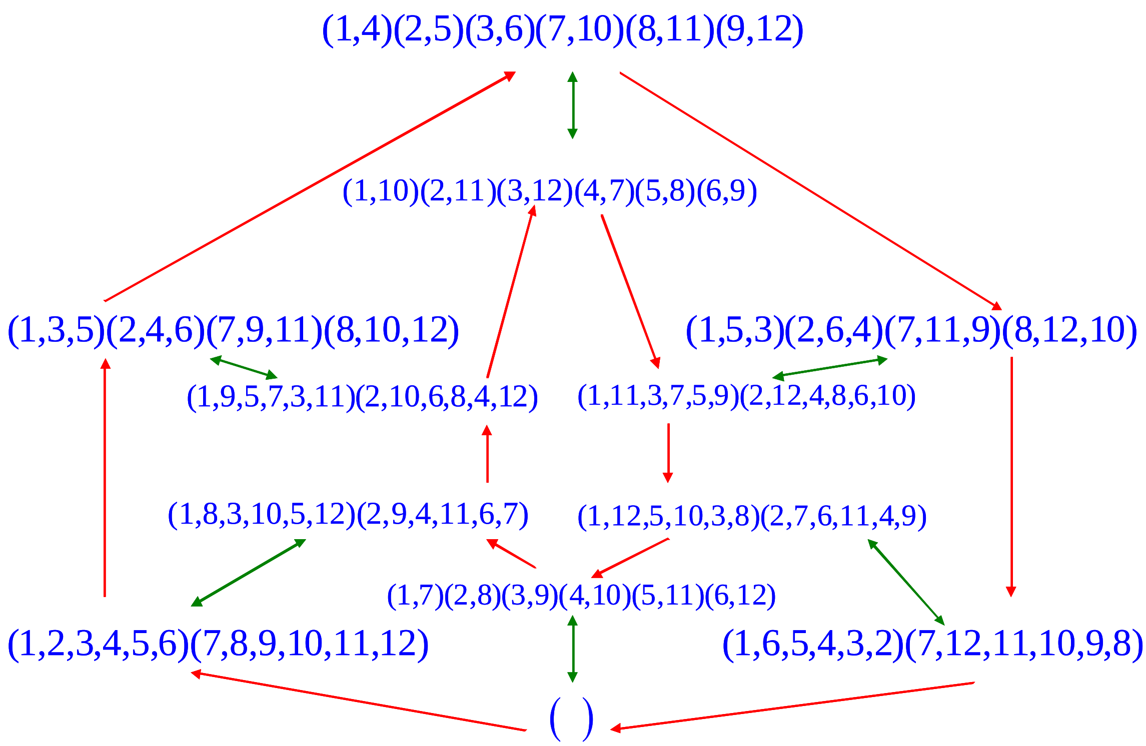
Cycle Graph


(1,4)(2,5)(3,6)
(1,4)(2,5)(3,6)(7,8)
(7,8)
(1,3,5)(2,4,6)
(1,5,3)(2,6,4)
(1,5,3)(2,6,4)(7,8)
(1,3,5)(2,4,6)(7,8)
(1,2,3,4,5,6)(7,8)
(1,6,5,4,3,2)(7,8)
(1,2,3,4,5,6)
(1,6,5,4,3,2)
( )

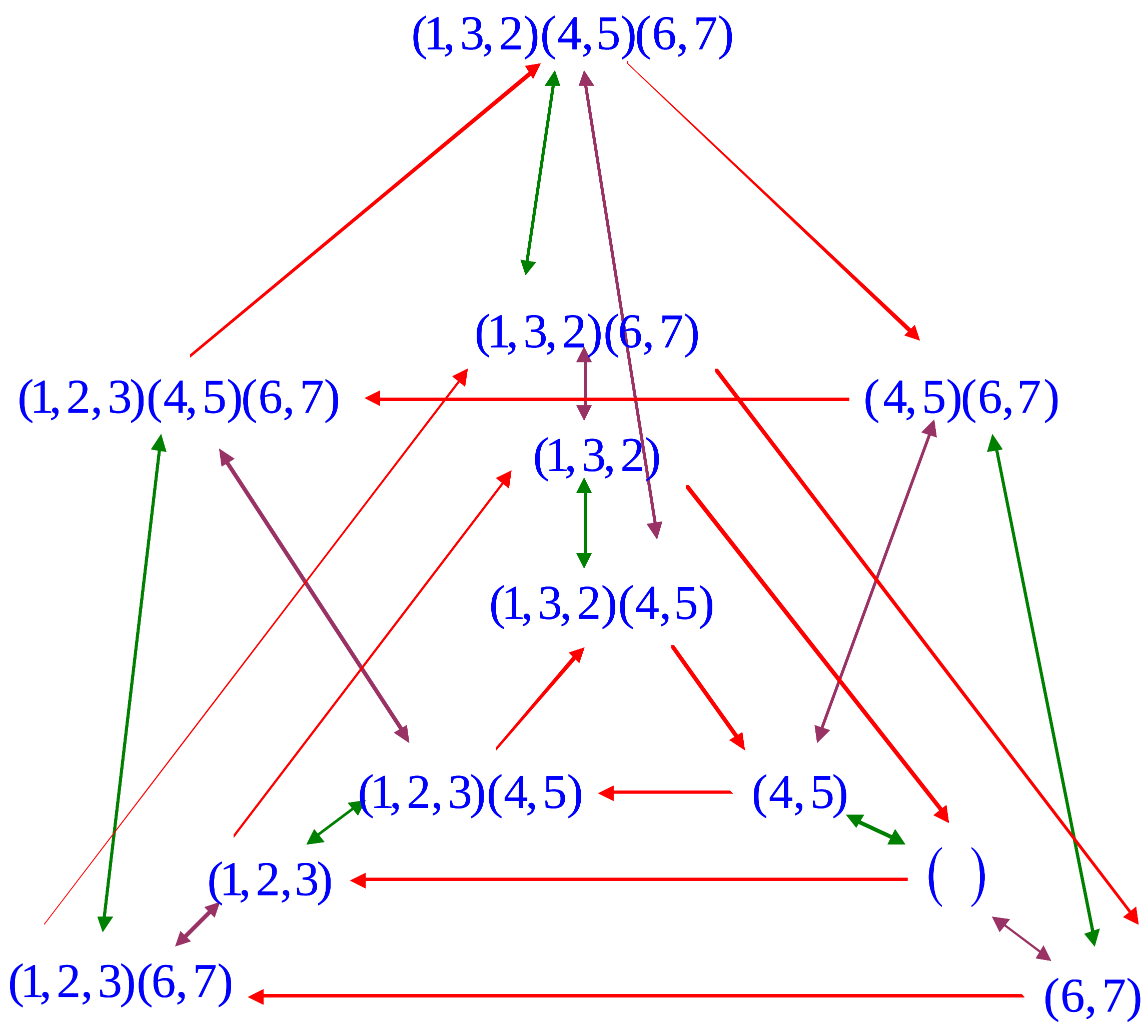
Cayley Diagram



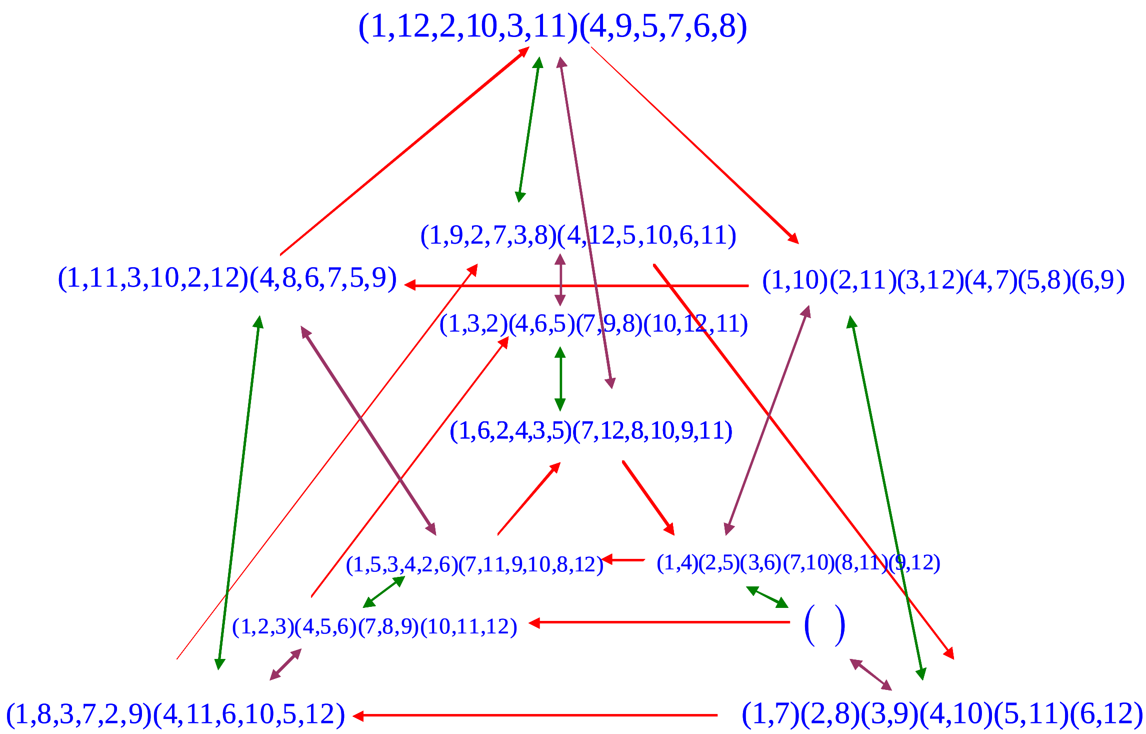
or


(1,4)(2,5)(3,6)(7,10)(8,11)(9,12)
(1,10)(2,11)(3,12)(4,7)(5,8)(6,9)
(1,3,5)(2,4,6)(7,9,11)(8,10,12)
(1,5,3)(2,6,4)(7,11,9)(8,12,10)
(1,9,5,7,3,11)(2,10,6,8,4,12)
(1,11,3,7,5,9)(2,12,4,8,6,10)
(1,8,3,10,5,12)(2,9,4,11,6,7)
(1,12,5,10,3,8)(2,7,6,11,4,9)
(1,7)(2,8)(3,9)(4,10)(5,11)(6,12)
(1,2,3,4,5,6)(7,8,9,10,11,12)
(1,6,5,4,3,2)(7,12,11,10,9,8)
( )

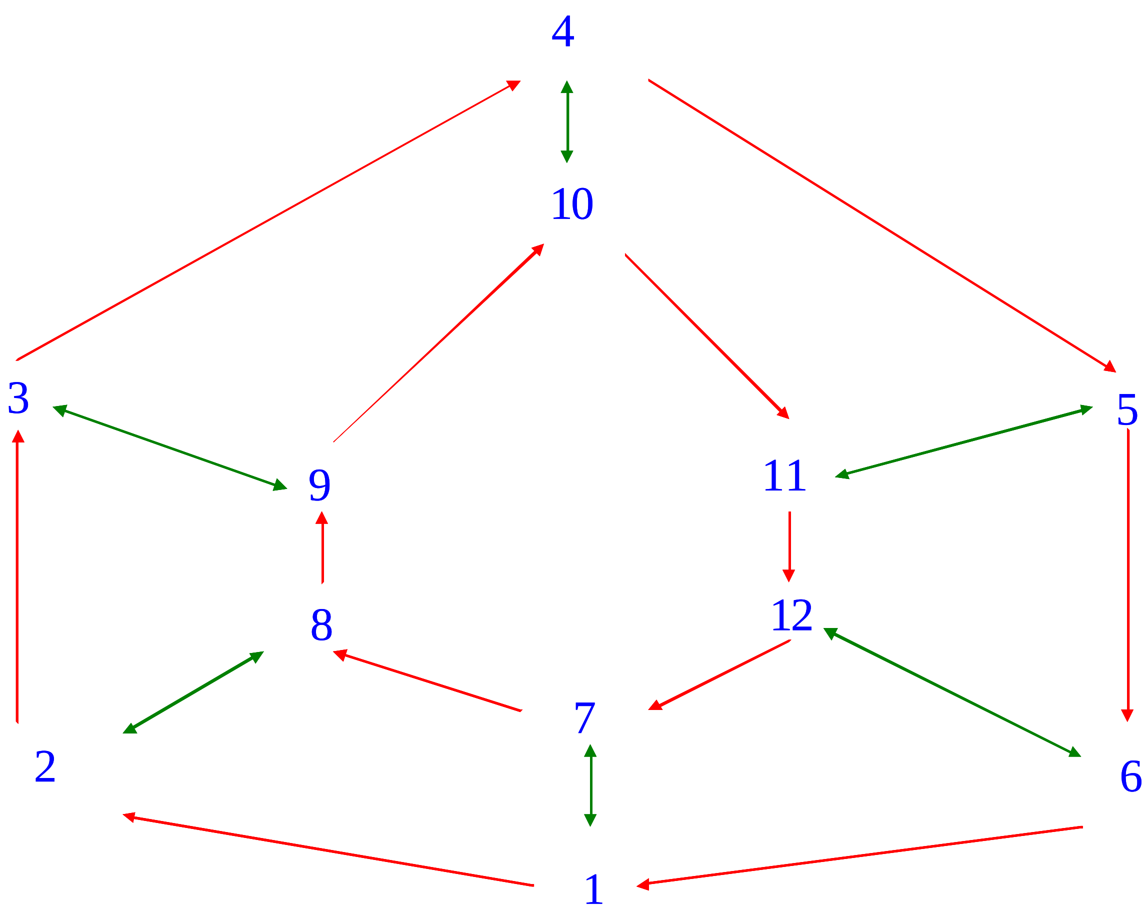
or



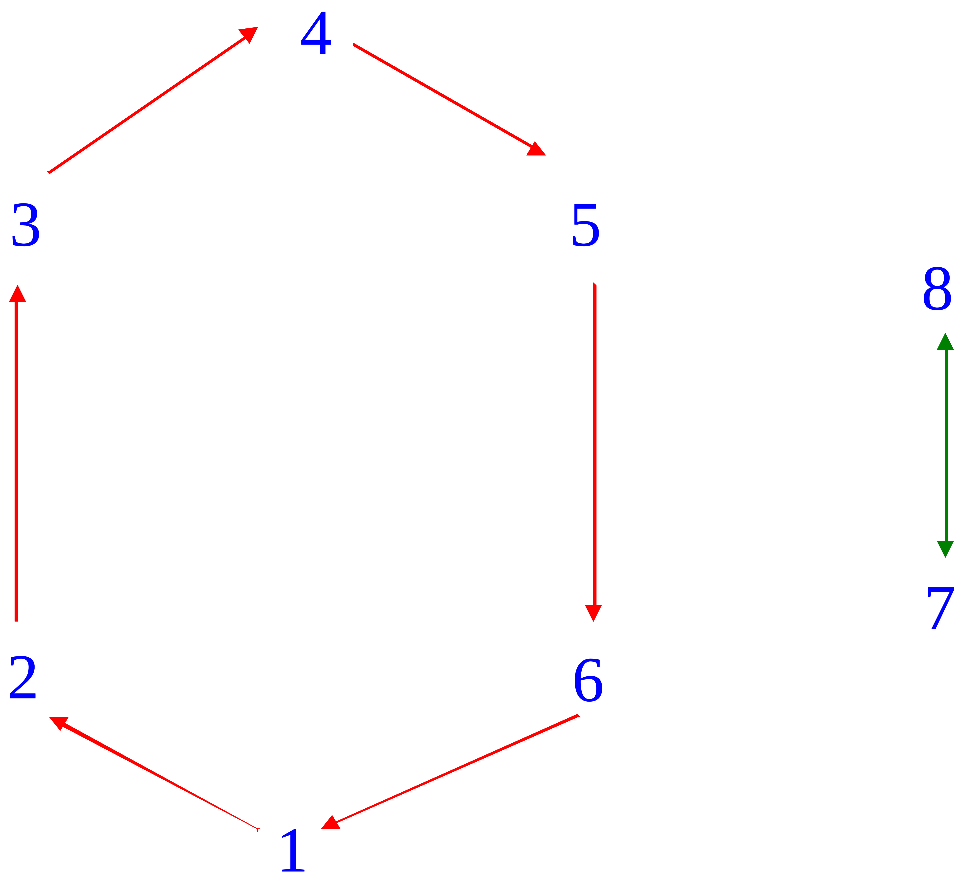
or


(1,12,2,10,3,11)(4,9,5,7,6,8)
(1,9,2,7,3,8)(4,12,5,10,6,11)
(1,11,3,10,2,12)(4,8,6,7,5,9)
(1,10)(2,11)(3,12)(4,7)(5,8)(6,9)
(1,3,2)(4,6,5)(7,9,8)(10,12,11)
(1,6,2,4,3,5)(7,12,8,10,9,11)
(1,5,3,4,2,6)(7,11,9,10,8,12)
(1,4)(2,5)(3,6)(7,10)(8,11)(9,12)
(1,2,3)(4,5,6)(7,8,9)(10,11,12)
( )
(1,8,3,7,2,9)(4,11,6,10,5,12)
(1,7)(2,8)(3,9)(4,10)(5,11)(6,12)

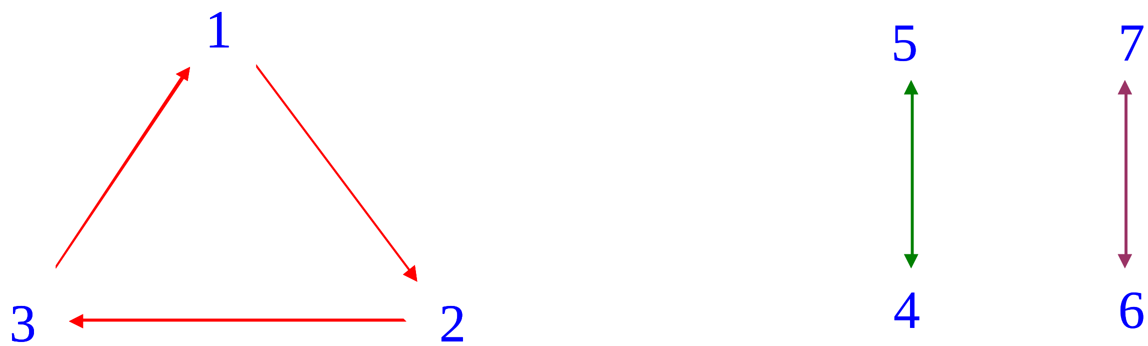
Generator Diagram



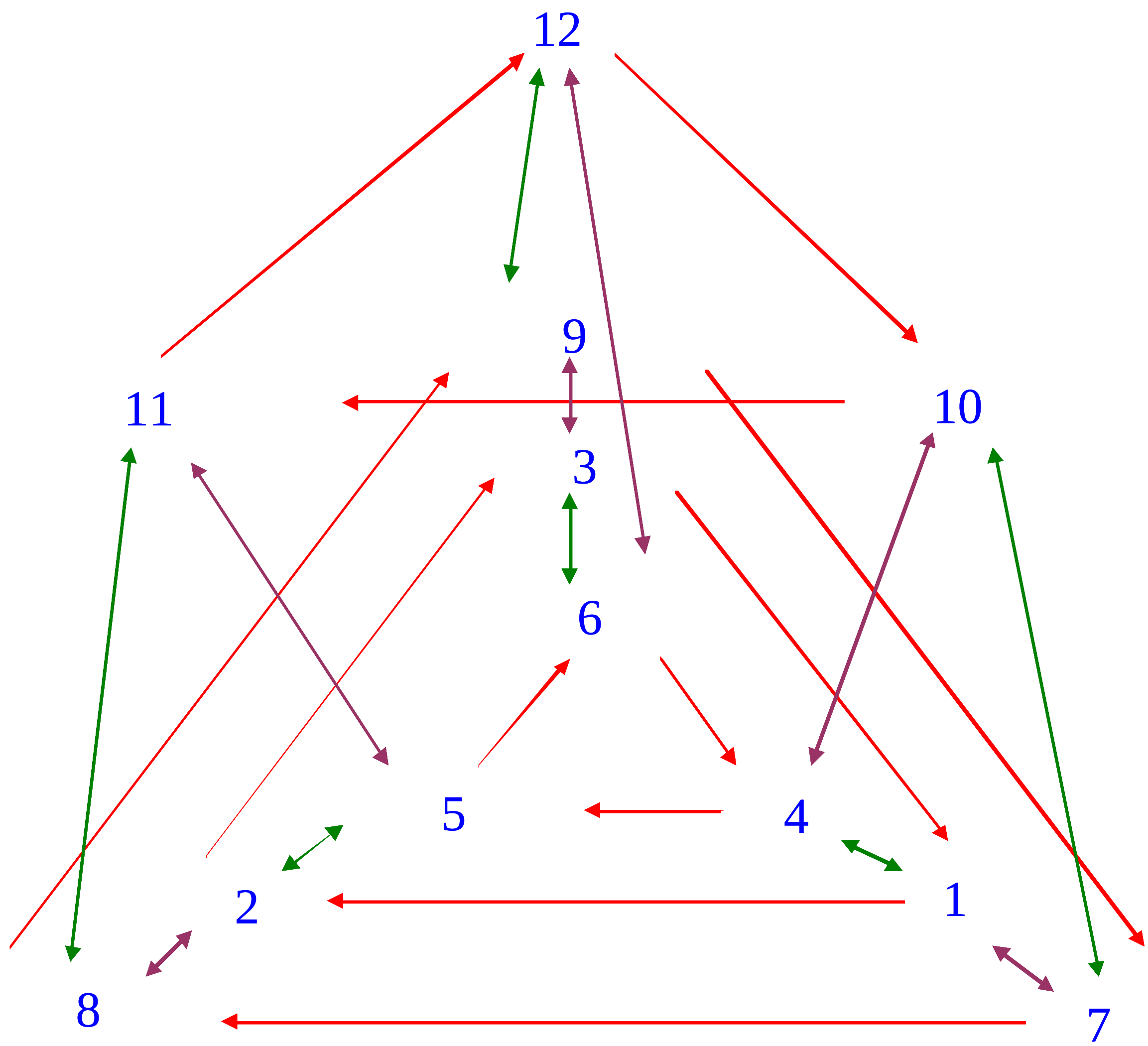
or



or



or



12
9
11
10
3
6
5
4
2
1
8
7

3. *Group*: $D_6 \cong \mathbb{Z}_6 >\!\triangleleft\, \mathbb{Z}_2 \cong C_6 >\!\triangleleft\, C_2$ (*dihedral group*)

Generators: $(1,2,3,4,5,6),(2,6)(3,5)$

THE *DIHEDRAL GROUP* D_6

$$D_6 \cong \mathbb{Z}_6 \rtimes \mathbb{Z}_2 \cong C_6 \rtimes C_2$$

Generators:

$(1,2,3,4,5,6),(2,6)(3,5)$

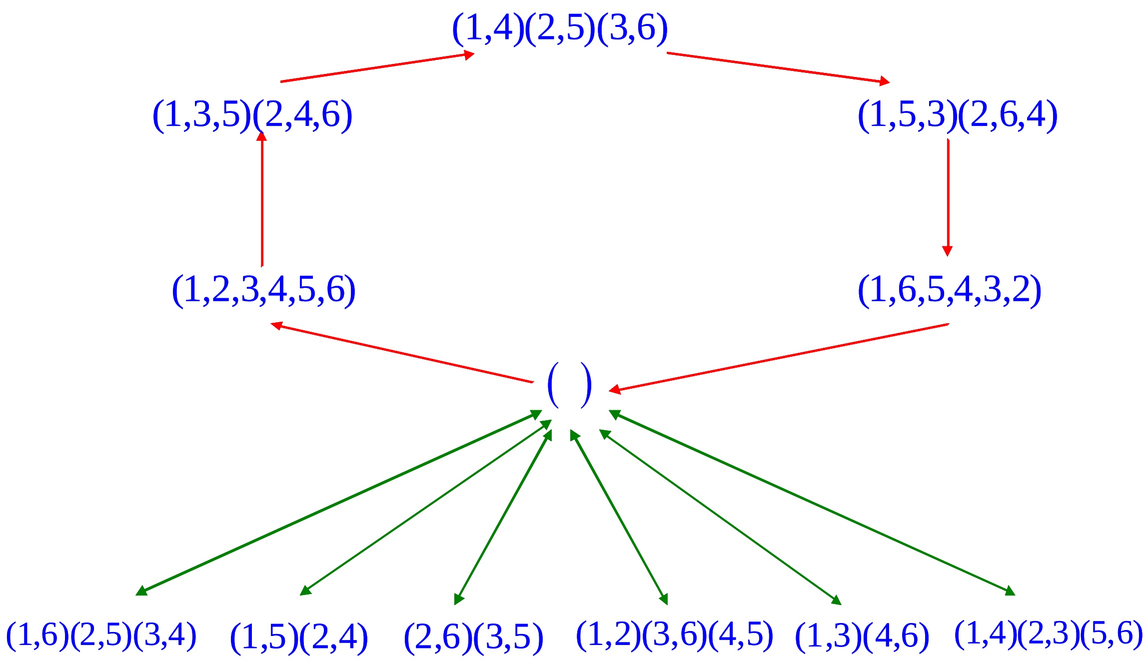
Elements:

$$\left\{\begin{array}{c} (\ ) \\ (2,6)(3,5) \\ (1,2)(3,6)(4,5) \\ (1,2,3,4,5,6) \\ (1,3)(4,6) \\ (1,3,5)(2,4,6) \\ (1,4)(2,3)(5,6) \\ (1,4)(2,5)(3,6) \\ (1,5)(2,4) \\ (1,5,3)(2,6,4) \\ (1,6,5,4,3,2) \\ (1,6)(2,5)(3,4) \end{array}\right\} \cong D_6$$

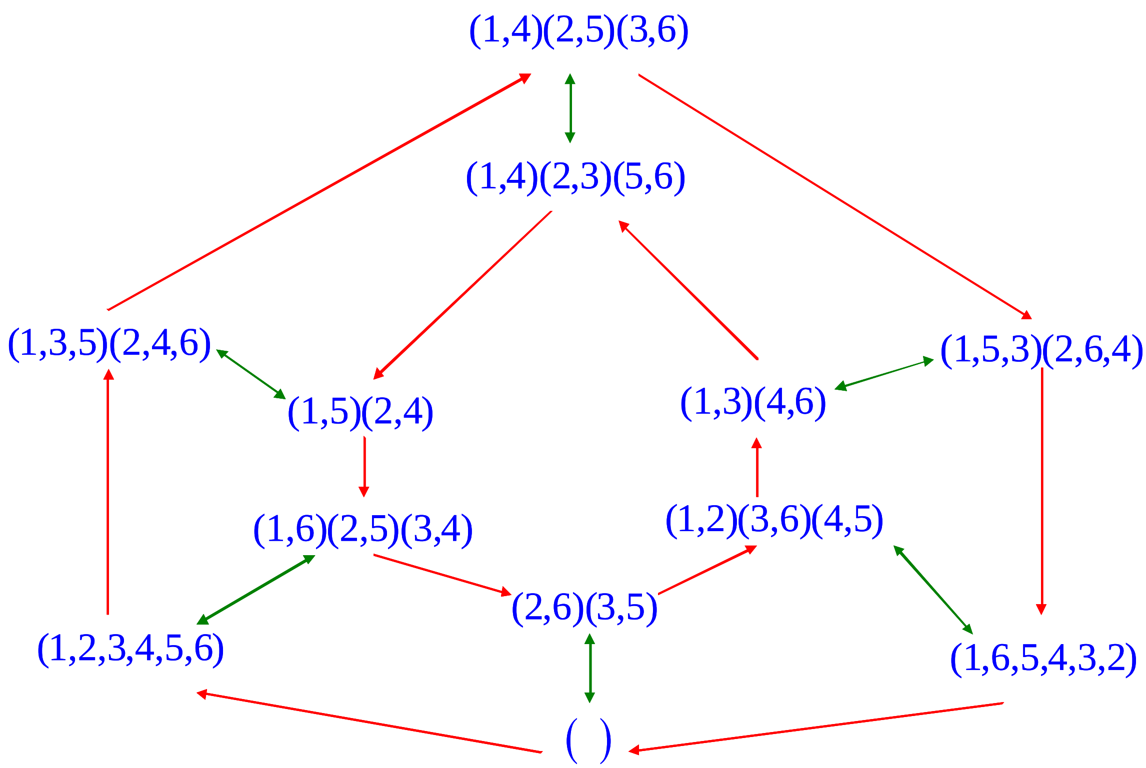
Is *Abelian*?

No

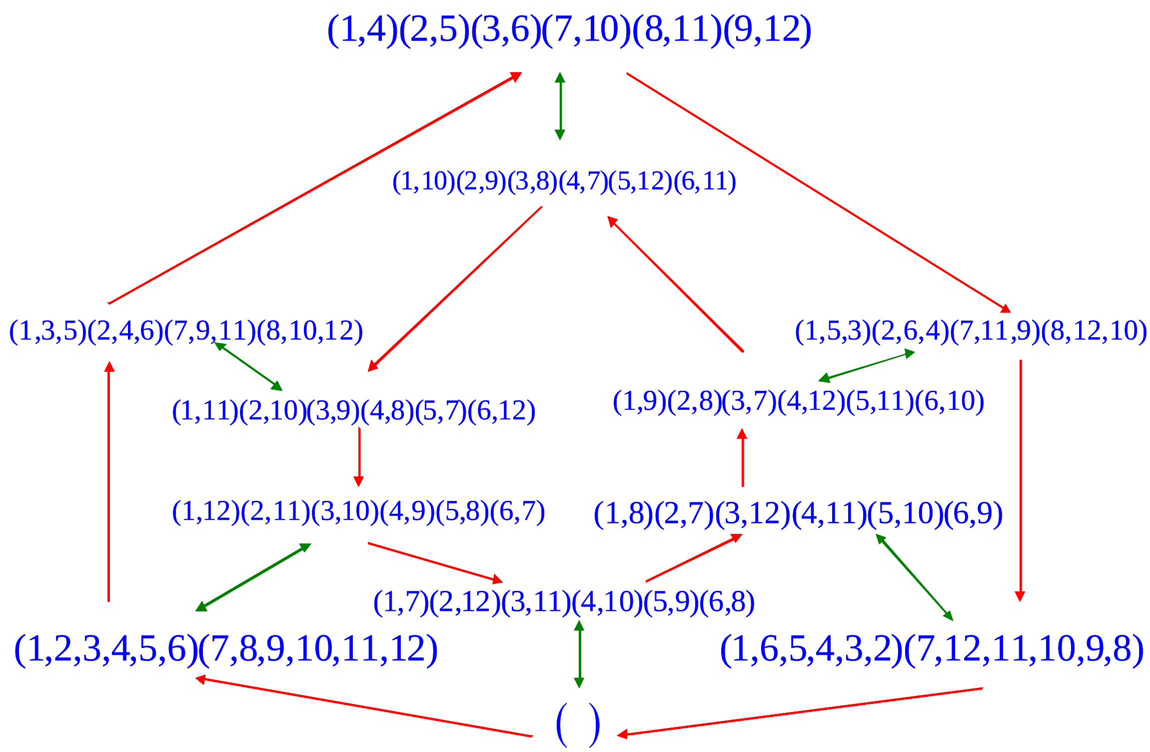
Cycle Graph


(1,4)(2,5)(3,6)
(1,3,5)(2,4,6)
(1,5,3)(2,6,4)
(1,2,3,4,5,6)
(1,6,5,4,3,2)
( )
(1,6)(2,5)(3,4)
(1,5)(2,4)
(2,6)(3,5)
(1,2)(3,6)(4,5)
(1,3)(4,6)
(1,4)(2,3)(5,6)

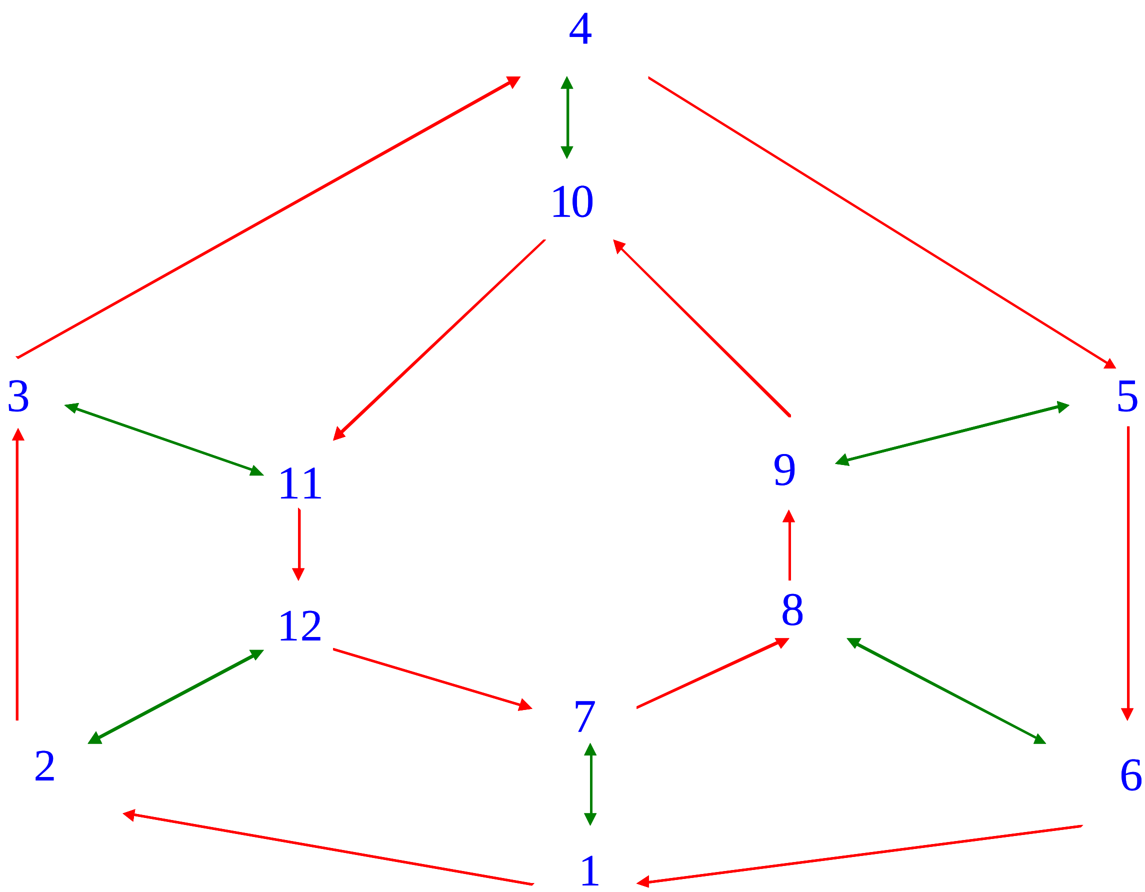
Cayley Diagram



or


(1,4)(2,5)(3,6)(7,10)(8,11)(9,12)
(1,10)(2,9)(3,8)(4,7)(5,12)(6,11)
(1,3,5)(2,4,6)(7,9,11)(8,10,12)
(1,5,3)(2,6,4)(7,11,9)(8,12,10)
(1,11)(2,10)(3,9)(4,8)(5,7)(6,12)
(1,9)(2,8)(3,7)(4,12)(5,11)(6,10)
(1,12)(2,11)(3,10)(4,9)(5,8)(6,7)
(1,8)(2,7)(3,12)(4,11)(5,10)(6,9)
(1,7)(2,12)(3,11)(4,10)(5,9)(6,8)
(1,2,3,4,5,6)(7,8,9,10,11,12)
(1,6,5,4,3,2)(7,12,11,10,9,8)
( )

Generator Diagram



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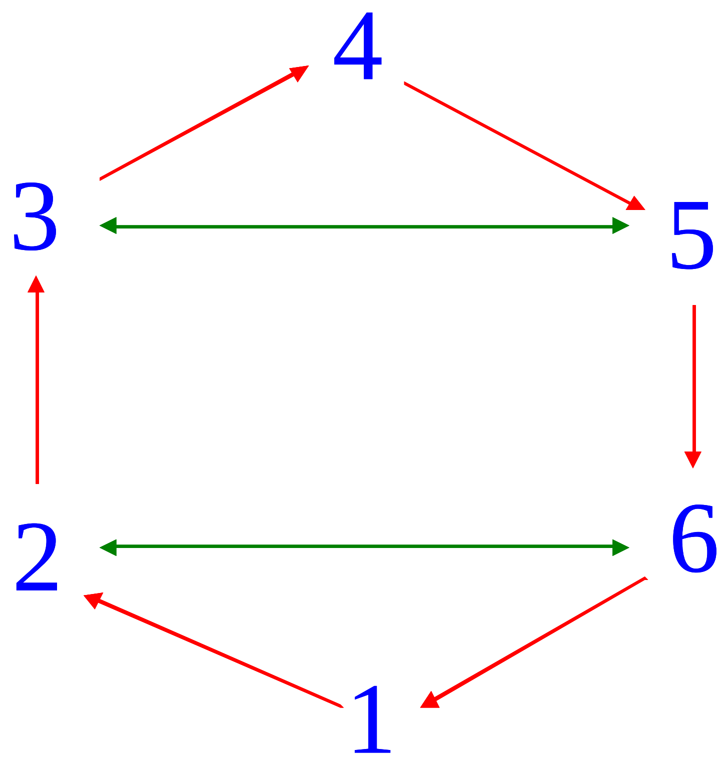

4
3
5
2
6
1

4. *Group*: A_4 (*alternating group*)

Generators: $(1,2,3),(2,3,4)$

THE *ALTERNATING GROUP* OF DEGREE 4 A_4

A_4

Generators:

$(1,2,3),(2,3,4)$

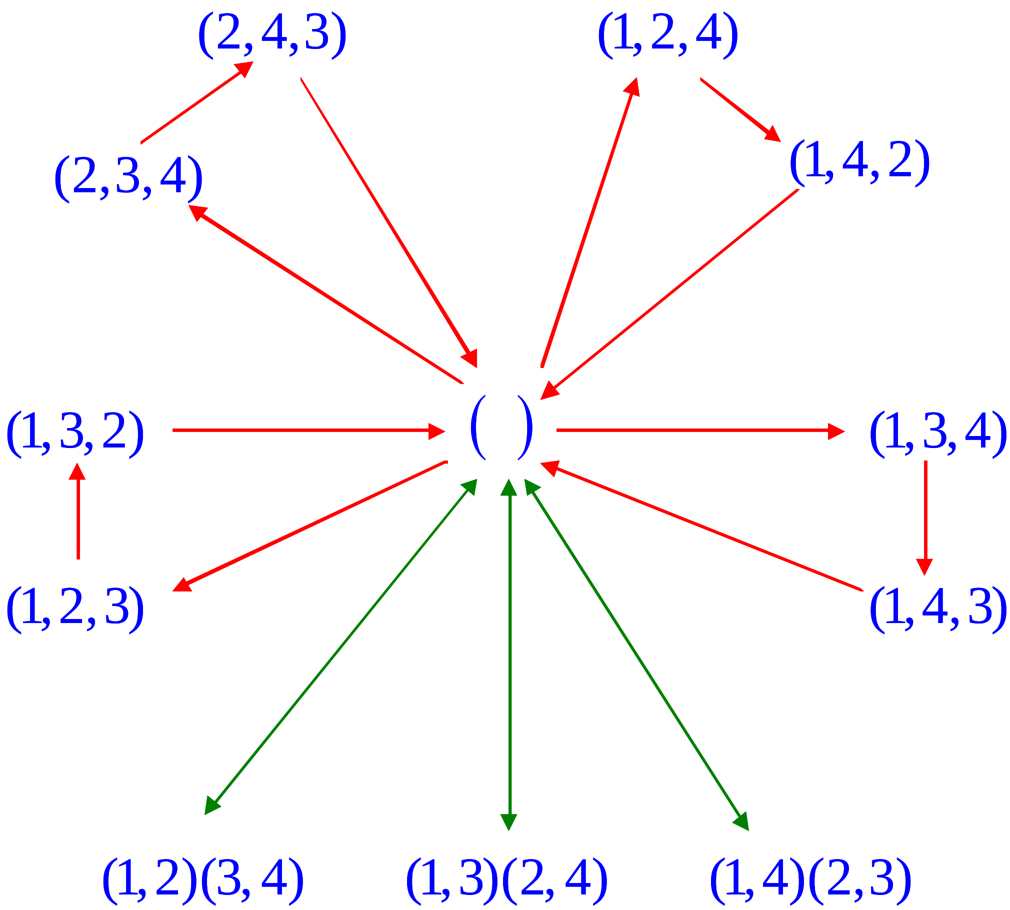
Elements:

$$\left\{\begin{array}{c} (\ ) \\ (2,3,4) \\ (2,4,3) \\ (1,2)(3,4) \\ (1,2,3) \\ (1,2,4) \\ (1,3,2) \\ (1,3,4) \\ (1,3)(2,4) \\ (1,4,2) \\ (1,4,3) \\ (1,4)(2,3) \end{array}\right\} \cong A_4$$

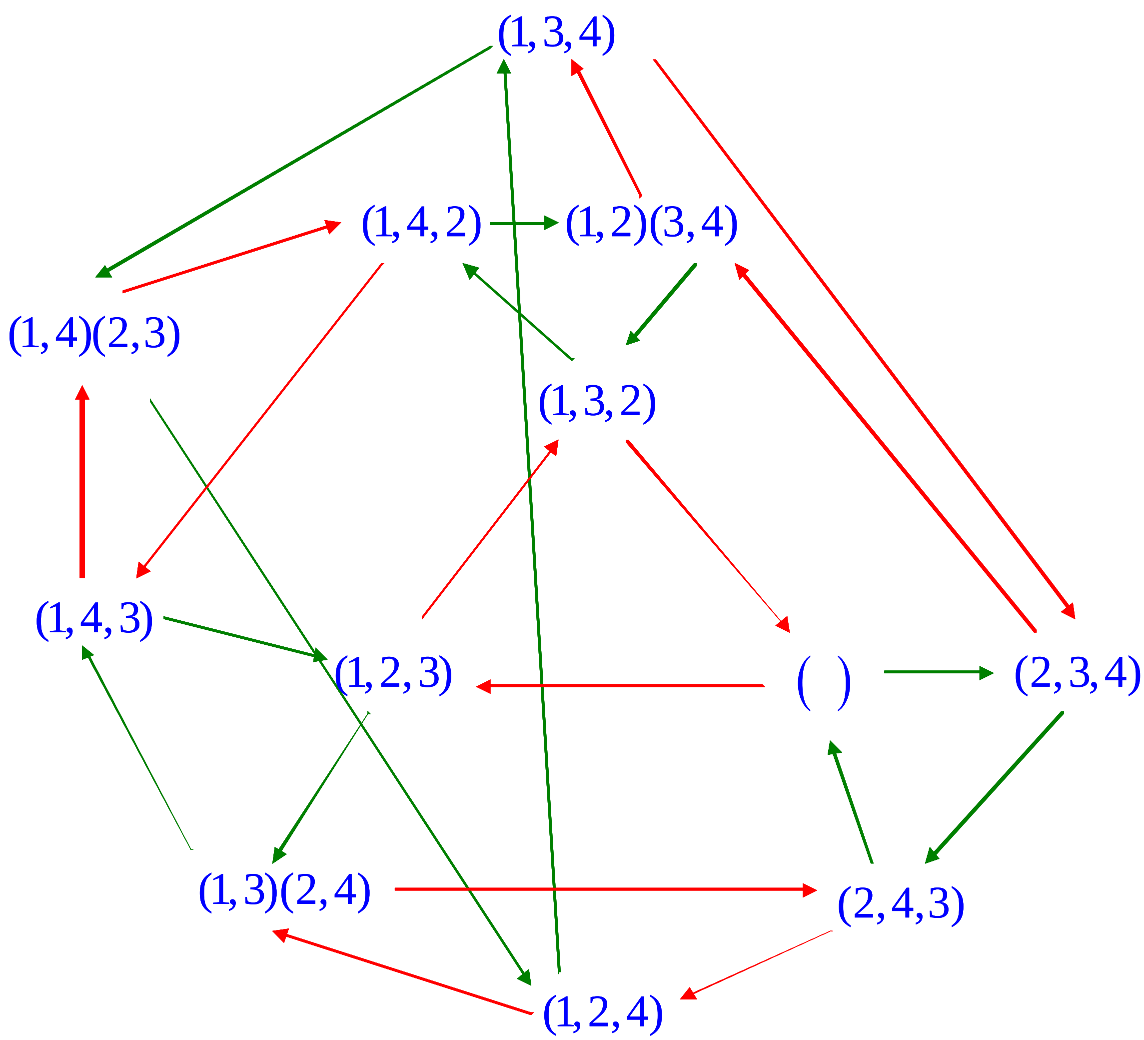
Is *Abelian*?

No

Cycle Graph



Cayley Diagram



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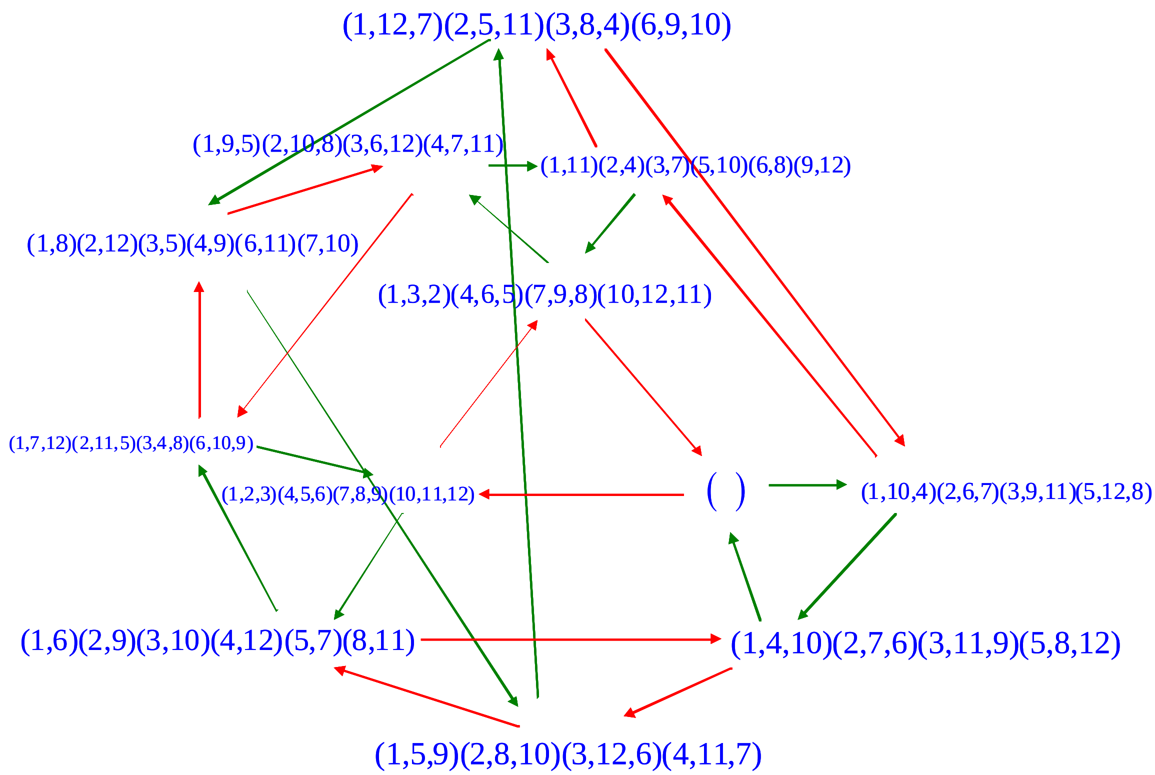
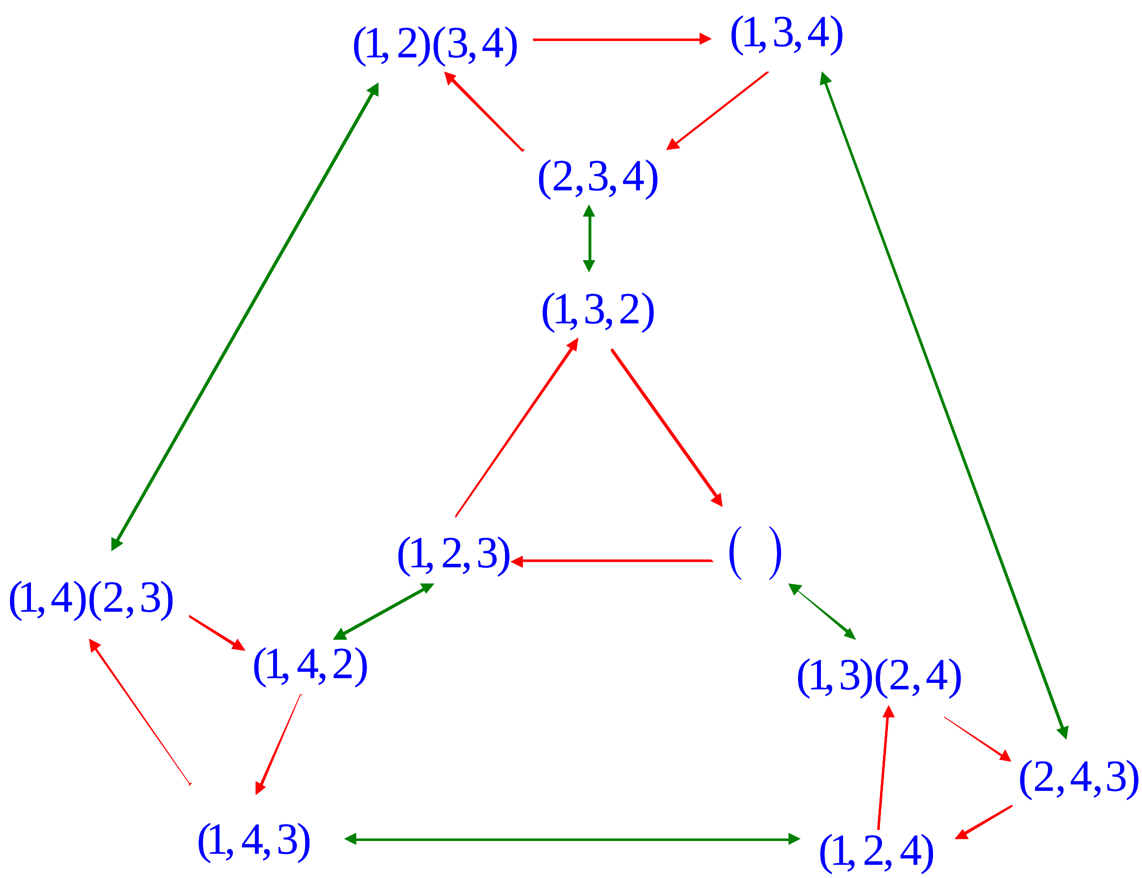
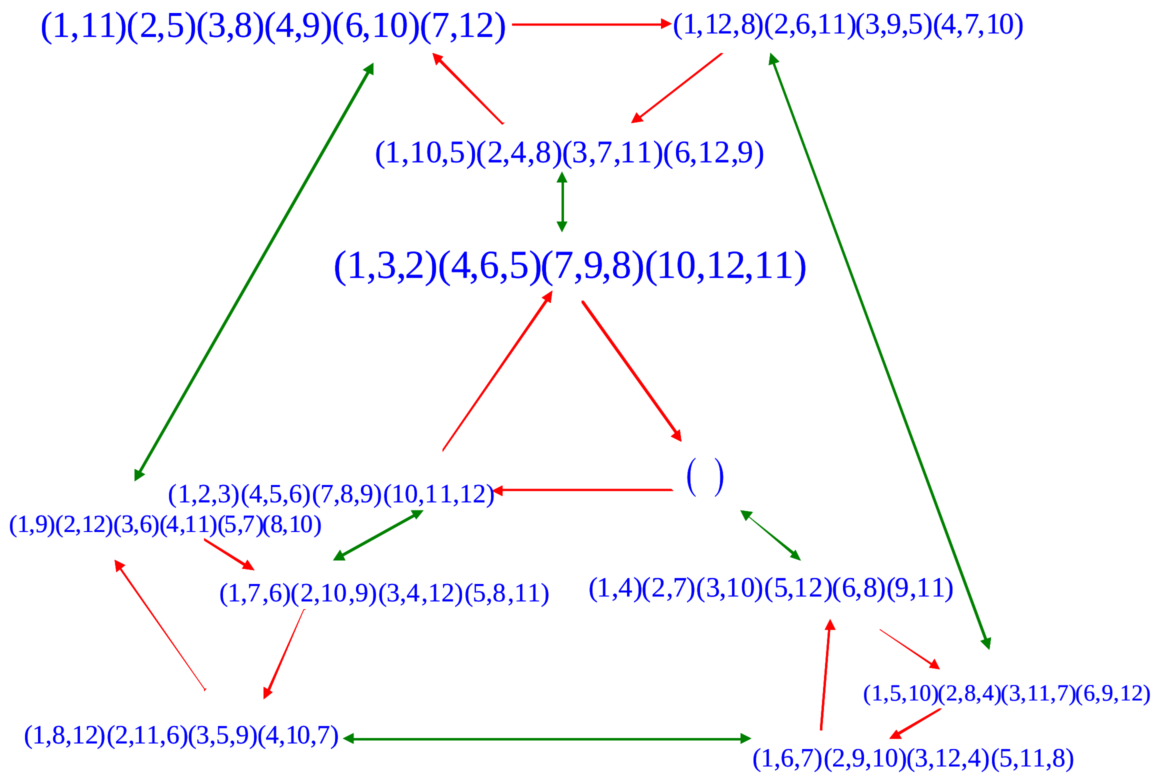

(1,12,7)(2,5,11)(3,8,4)(6,9,10)
(1,9,5)(2,10,8)(3,6,12)(4,7,11)
(1,11)(2,4)(3,7)(5,10)(6,8)(9,12)
(1,8)(2,12)(3,5)(4,9)(6,11)(7,10)
(1,3,2)(4,6,5)(7,9,8)(10,12,11)
(1,7,12)(2,11,5)(3,4,8)(6,10,9)
(1,2,3)(4,5,6)(7,8,9)(10,11,12)
( )
(1,10,4)(2,6,7)(3,9,11)(5,12,8)
(1,6)(2,9)(3,10)(4,12)(5,7)(8,11)
(1,4,10)(2,7,6)(3,11,9)(5,8,12)
(1,5,9)(2,8,10)(3,12,6)(4,11,7)

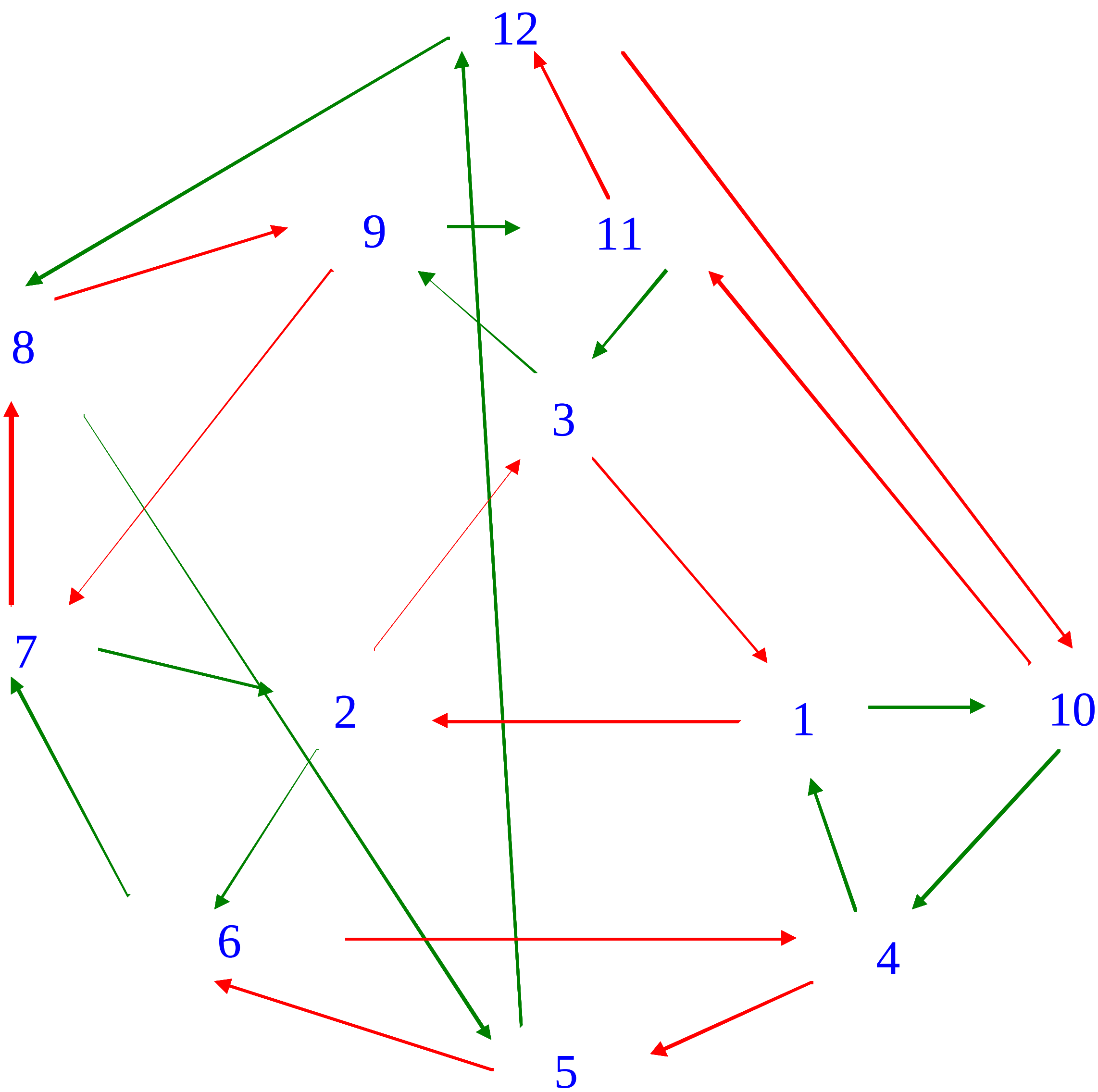
or



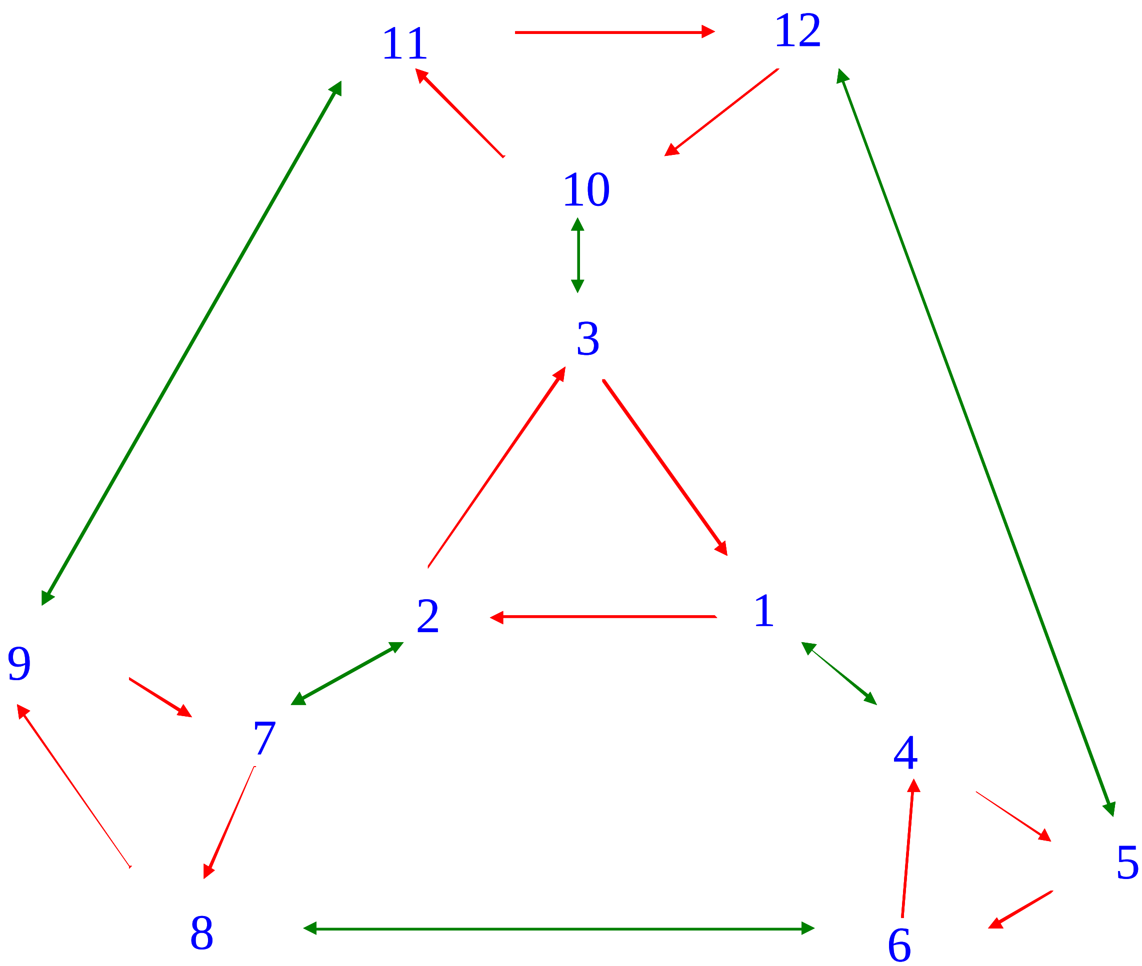
or


(1,11)(2,5)(3,8)(4,9)(6,10)(7,12)
(1,12,8)(2,6,11)(3,9,5)(4,7,10)
(1,10,5)(2,4,8)(3,7,11)(6,12,9)
(1,3,2)(4,6,5)(7,9,8)(10,12,11)
(1,2,3)(4,5,6)(7,8,9)(10,11,12)
( )
(1,9)(2,12)(3,6)(4,11)(5,7)(8,10)
(1,7,6)(2,10,9)(3,4,12)(5,8,11)
(1,4)(2,7)(3,10)(5,12)(6,8)(9,11)
(1,5,10)(2,8,4)(3,11,7)(6,9,12)
(1,8,12)(2,11,6)(3,5,9)(4,10,7)
(1,6,7)(2,9,10)(3,12,4)(5,11,8)

Generator Diagram



or


11
12
10
3
2
1
9
7
4
5
8
6

YOUR KNOWLEDGE OF GROUP THEORY
IS COMING ALONG SWIMMINGLY!