A CHILD'S GARDEN OF GROUPS

The Structure of Groups of Order 1 through 10

(Part 4)

by

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INTRODUCTION (PART 4)

We can never learn about all the numbers in our number system since there are an infinite number of them and we have only a finite amount of time to live. Nonetheless, because we know multiplication tables and lots of other facts about many small numbers, we feel that we know numbers in general and we have a good feel for what is true and what is not true about numbers. This same strategy can also be applied to the study of *group theory*. We can never know all there is to know about every single *group*, but if we study several *groups* of small order in detail, then that will gives us a good base of knowledge to draw upon when thinking about *groups*. Thus, in this part of our work we'll examine all *groups* of orders 1 through 10, and almost all of them will be either *cyclic groups*, *dihedral groups*, *symmetric groups*, *alternating groups, direct products*, or *semidirect products*.. More specifically, we'll identify permutations that generate each *group*, list the elements of the *group*, identify whether the *group* is *abelian* or not, list important *subgroups* of each *group*, draw the what's known as a *subgroup lattice* (a diagram that visually illustrates the *subgroup* structure of a *group*), and we'll identify whether each *subgroup* is *normal* or not. Also, if a *subgroup H* of a *group G* is not *normal* in *G* and if $a \in G$, then both aHa^{-1} and $a^{-1}Ha$ are also *subgroups* of *G* called *conjugates* of *H* in *G* (see Part 2). Furthermore, if a *subgroup* is not *normal*, then we'll express that *subgroup* and its *conjugates* all in the same color in the charts that follow. Also, just as a *subgroup* can have *conjugates* by an element, so can we find the *conjugate* of a single element. Thus, if we have a *subgroup H* of a *group G* and if $a \in H$ and $b \in G$, then both bab^{-1} and $b^{-1}ab$ are called *conjugates* of *a* by *b*. Also, GAP software will define a^b as meaning $b^{-1}ab$, but be aware that some authors define a^b as meaning bab^{-1}. Additionally, if our *group* is *abelian*, then it's pretty easy to identify its internal structure thanks to the <u>*Fundamental Theorem of Finite Abelain Groups*</u>. This theorem tells us that <u>*every finite abelian group is a direct*</u>

product of groups of prime power order. Thus, for example, the only possible *abelian* groups of order 8 are C_8, $C_4 \times C_2 \cong C_2 \times C_4$, and $C_2 \times C_2 \times C_2$. And lastly, just because a *group* has small order (a small number of elements), never assume that you have exhausted everything you can learn about it. And this goes even for the *identity group*. He who has contemplated the *identity group* 100 times never knows as much as he who has contemplated it 101 times! Enjoy!

HOW TO USE GAP (PART 4)

In Part 4 of *How to Use GAP*, we've added on to the very end of our list some commands (in red) that will help you find *conjugates* of various sorts.

1. *How can I redisplay the previous command in order to edit it?*

   Press down on the control key and then also press p. In other words, "Ctrl p".

2. *If the program gets in a loop and shows you the prompt "brk>" instead of "gap>", how can I exit the loop?*

   Press down on the control key and then also press d. In other words, "Ctrl d".

3. *How can I exit the program?*

   Either click on the "close" box for the window, or type "quit;" and press "Enter."

4. *How do I find the inverse of a permutation?*

   gap> a:=(1,2,3,4);
   (1,2,3,4)
   gap> a^-1;
   (1,4,3,2)

5. *How can I multiply permutations and raise permutations to powers?*

```
gap> (1,2)*(1,2,3);
(1,3)

gap> (1,2,3)^2;
(1,3,2)

gap> (1,2,3)^-1;
(1,3,2)

gap> (1,2,3)^-2;
(1,2,3)

gap> a:=(1,2,3);
(1,2,3)

gap> b:=(1,2);
(1,2)

gap> a*b;
(2,3)

gap> a^2;
(1,3,2)

gap> a^-2;
(1,2,3)

gap> a^3;
()
```

```
gap> a^-3;
()

gap> (a*b)^2;
()

gap> (a*b)^3;
(2,3)
```

6. *How can I create a group from permutations, find the size of the group, and find the elements in the group?*

```
gap> a:=(1,2);
(1,2)

gap> b:=(1,2,3);
(1,2,3)

gap> g1:=Group(a,b);
Group([ (1,2), (1,2,3) ])

gap> Size(g1);
6

gap> Elements(g1);
[ (), (2,3), (1,2), (1,2,3), (1,3,2), (1,3) ]

gap> g2:=Group([(1,2),(1,2,3)]);
Group([ (1,2), (1,2,3) ])
```

```
gap> g3:=Group((1,2),(2,3,4));
Group([ (1,2), (2,3,4) ])
```

7. *How can I create a cyclic group of order 3?*

```
gap> a:=(1,2,3);
(1,2,3)

gap> g1:=Group(a);
Group([ (1,2,3) ])

gap> Size(g1);
3

gap> Elements(g1);
[ (), (1,2,3), (1,3,2) ]

gap> g2:=Group((1,2,3));
Group([ (1,2,3) ])

gap> g3:=CyclicGroup(IsPermGroup,3);
Group([ (1,2,3) ])
```

8. *How can I create a multiplication table for the cyclic group of order 3 that I just created?*

```
gap> ShowMultiplicationTable(g1);

*       | ()       (1,2,3)  (1,3,2)
--------+--------------------------------
()      | ()       (1,2,3)  (1,3,2)
(1,2,3) | (1,2,3)  (1,3,2)  ()
(1,3,2) | (1,3,2)  ()       1,2,3)
```

9. *How do I determine if a group is abelian?*

```
gap> g1:=Group((1,2,3));
Group([ (1,2,3) ])

gap> IsAbelian(g1);
true

gap> g2:=Group((1,2),(1,2,3));
Group([ (1,2), (1,2,3) ])

gap> IsAbelian(g2);
false
```

10. *What do I type in order to get help for a command like "Elements?"*

```
gap> ?Elements
```

11. *How do I find all subgroups of a group?*

```
gap> a:=(1,2,3);
(1,2,3)
```

```
gap> b:=(2,3);
(2,3)

gap> g:=Group(a,b);
Group([ (1,2,3), (2,3) ])

gap> Size(g);
6

gap> Elements(g);
[ (), (2,3), (1,2), (1,2,3), (1,3,2), (1,3) ]

gap> h:=AllSubgroups(g);
[ Group(()), Group([ (2,3) ]), Group([ (1,2) ]), Group([ (1,3) ]),
Group([ (1,2,3) ]), Group([ (1,2,3), (2,3) ]) ]

gap> List(h,i->Elements(i));
[ [ () ], [ (), (2,3) ], [ (), (1,2) ], [ (), (1,3) ], [ (), (1,2,3),
(1,3,2) ], [ (), (2,3), (1,2), (1,2,3), (1,3,2), (1,3) ] ]

gap> Elements(h[1]);
[ () ]

gap> Elements(h[2]);
[ (), (2,3) ]

gap> Elements(h[3]);
[ (), (1,2) ]

gap> Elements(h[4]);
[ (), (1,3) ]


gap> Elements(h[5]);
[ (), (1,2,3), (1,3,2) ]

gap> Elements(h[6]);
[ (), (2,3), (1,2), (1,2,3), (1,3,2), (1,3) ]
```

12. *How do I find the subgroup generated by particular permutations?*

```
gap> g:=Group((1,2),(1,2,3));
Group([ (1,2), (1,2,3) ])

gap> Elements(g);
[ (), (2,3), (1,2), (1,2,3), (1,3,2), (1,3) ]

gap> h:=Subgroup(g,[(1,2)]);
Group([ (1,2) ])

gap> Elements(h);
[ (), (1,2) ]
```

13. *How do I determine if a subgroup is normal?*

```
gap> g:=Group((1,2),(1,2,3));
Group([ (1,2), (1,2,3) ])

gap> h1:=Group((1,2));
Group([ (1,2) ])
```

```
gap> IsNormal(g,h1);

gap> h2:=Group((1,2,3));
Group([ (1,2,3) ])


gap> IsNormal(g,h2);
true
```

14. *How do I find all normal subgroups of a group?*

```
gap> g:=Group((1,2),(1,2,3));
Group([ (1,2), (1,2,3) ])

gap> Elements(g);
[ (), (2,3), (1,2), (1,2,3), (1,3,2), (1,3) ]

gap> n:=NormalSubgroups(g);
[ Group([ (1,2), (1,2,3) ]), Group([ (1,3,2) ]), Group(()) ]

gap> Elements(n[1]);
[ (), (2,3), (1,2), (1,2,3), (1,3,2), (1,3) ]

gap> Elements(n[2]);
[ (), (1,2,3), (1,3,2) ]

gap> Elements(n[3]);
[ () ]
```

15. *How do I determine if a group is simple?*

```
gap> g:=Group((1,2),(1,2,3));
Group([ (1,2), (1,2,3) ])

gap> Elements(g);
[ (), (2,3), (1,2), (1,2,3), (1,3,2), (1,3) ]

gap> IsSimple(g);
false


gap> h:=Group((1,2));
Group([ (1,2) ])

gap> Elements(h);
[ (), (1,2) ]

gap> IsSimple(h);
true
```

16. *How do I find the right cosets of a subset H of G?*

```
gap> g:=Group([(1,2,3),(1,2)]);
Group([ (1,2,3), (1,2) ])

gap> Elements(g);
[ (), (2,3), (1,2), (1,2,3), (1,3,2), (1,3) ]

gap> h:=Subgroup(g,[(1,2)]);
Group([ (1,2) ])

gap> Elements(h);
[ (), (1,2) ]

gap> c:=RightCosets(g,h);
[ RightCoset(Group( [ (1,2) ] ),()), RightCoset(Group( [ (1,2) ] ),(1,3,2)),
RightCoset(Group( [ (1,2) ] ),(1,2,3)) ]

gap> List(c,i->Elements(i));
[ [ (), (1,2) ], [ (2,3), (1,3,2) ], [ (1,2,3), (1,3) ] ]
gap> Elements(c[1]);
[ (), (1,2) ]

gap> Elements(c[2]);
[ (2,3), (1,3,2) ]


gap> Elements(c[3]);
[ (1,2,3), (1,3) ]

gap> rc:=RightCoset(h,(1,2,3));
RightCoset(Group([ (1,2) ]),(1,2,3))

gap> Elements(rc);
[ (1,2,3), (1,3) ]

gap> rc:=h*(1,2,3);
RightCoset(Group([ (1,2) ]),(1,2,3))

gap> Elements(rc);
[ (1,2,3), (1,3) ]
```

17. *How can I create a quotient (factor) group?*

```
gap> g:=Group([(1,2,3),(1,2)]);
Group([ (1,2,3), (1,2) ])

gap> Elements(g);
[ (), (2,3), (1,2), (1,2,3), (1,3,2), (1,3) ]

gap> n:=Group((1,2,3));
Group([ (1,2,3) ])

gap> Elements(n);
[ (), (1,2,3), (1,3,2) ]

gap> IsNormal(g,n);
true

gap> c:=RightCosets(g,n);
[ RightCoset(Group([ (1,2,3) ]),()), RightCoset(Group([ (1,2,3) ]),(2,3)) ]
```

```
gap> Elements(c[1]);
[ (), (1,2,3), (1,3,2) ]

gap> Elements(c[2]);
[ (2,3), (1,2), (1,3) ]

gap> f:=FactorGroup(g,n);
Group([ f1 ])

gap> Elements(f);
[ <identity> of ..., f1 ]

gap> ShowMultiplicationTable(f);
*                  | <identity> of ... f1
-------------------+-----------------------------------
<identity> of ... | <identity> of ... f1
f1                 | f1                <identity> of ...
```

18. *How do I find the center of a group?*

```
gap> a:=(1,2,3);
(1,2,3)

gap> b:=(2,3);
(2,3)

gap> g:=Group(a,b);
Group([ (1,2,3), (2,3) ])

gap> Center(g);
Group(())

gap> c:=Center(g);
Group(())

gap> Elements(c);
[ () ]


gap> a:=(1,2,3,4);
(1,2,3,4)

gap> b:=(1,3);
(1,3)

gap> g:=Group(a,b);
Group([ (1,2,3,4), (1,3) ])

gap> c:=Center(g);
Group([ (1,3)(2,4) ])

gap> Elements(c);
[ (), (1,3)(2,4) ]
```

19. *How do I find the commutator (derived) subgroup of a group?*

```
gap> a:=(1,2,3);
(1,2,3)
```

```
gap> b:=(2,3);
(2,3)

gap> g:=Group(a,b);
Group([ (1,2,3), (2,3) ])

gap> d:=DerivedSubgroup(g);
Group([ (1,3,2) ])

gap> Elements(d);
[ (), (1,2,3), (1,3,2) ]


gap> a:=(1,2,3,4);
(1,2,3,4)

gap> b:=(1,3);
(1,3)

gap> g:=Group(a,b);
Group([ (1,2,3,4), (1,3) ])

gap> d:=DerivedSubgroup(g);
Group([ (1,3)(2,4) ])

gap> Elements(d);
[ (), (1,3)(2,4) ]
```

20. *How do I find all Sylow p-subgroups for a given group?*

```
gap> a:=(1,2,3);
(1,2,3)

gap> b:=(2,3);
(2,3)

gap> g:=Group(a,b);
Group([ (1,2,3), (2,3) ])

gap> Size(g);
6

gap> FactorsInt(6);
[ 2, 3 ]

gap> sylow2:=SylowSubgroup(g,2);
Group([ (2,3) ])

gap> IsNormal(g,sylow2);
false

gap> c:=ConjugateSubgroups(g,sylow2);
[ Group([ (2,3) ]), Group([ (1,3) ]), Group([ (1,2) ]) ]

gap> Elements(c[1]);
[ (), (2,3) ]

gap> Elements(c[2]);
[ (), (1,3) ]

gap> Elements(c[3]);
[ (), (1,2) ]

gap> sylow3:=SylowSubgroup(g,3);
Group([ (1,2,3) ])
```

```
gap> IsNormal(g,sylow3);
true

gap> Elements(sylow3);
[ (), (1,2,3), (1,3,2) ]
```

21. *How can I create the Rubik's cube group using GAP?*

First you need to save the following permutations as a pure text file with the name rubik.txt to your C-drive before you can import it into GAP.

```
r:=(25,27,32,30)(26,29,31,28)(3,38,43,19)(5,36,45,21)(8,33,48,24);
l:=(9,11,16,14)(10,13,15,12)(1,17,41,40)(4,20,44,37)(6,22,46,35);
u:=(1,3,8,6)(2,5,7,4)(9,33,25,17)(10,34,26,18)(11,35,27,19);
d:=(41,43,48,46)(42,45,47,44)(14,22,30,38)(15,23,31,39)(16,24,32,40);
f:=(17,19,24,22)(18,21,23,20)(6,25,43,16)(7,28,42,13)(8,30,41,11);
b:=(33,35,40,38)(34,37,39,36)(3,9,46,32)(2,12,47,29)(1,14,48,27);
```

And now you can read the file into GAP and begin exploring.

```
gap> Read("C:/rubik.txt");

gap> rubik:=Group(r,l,u,d,f,b);
<permutation group with 6 generators>

gap> Size(rubik);
43252003274489856000
```

22. *How can I find the center of the Rubik's cube group?*

```
gap> c:=Center(rubik);
Group([ (2,34)(4,10)(5,26)(7,18)(12,37)(13,20)(15,44)(21,28)(23,42)(29,36)(31,4
5)(39,47) ])

gap> Size(c);
2

gap> Elements(c);
[ (), (2,34)(4,10)(5,26)(7,18)(12,37)(13,20)(15,44)(21,28)(23,42)(29,36)(31,45)
(39,47) ]
```

23. *How can I find the commutator (derived) subgroup of the Rubik's cube group?*

```
gap> d:=DerivedSubgroup(rubik);
<permutation group with 5 generators>

gap> Size(d);
21626001637244928000

gap> IsNormal(rubik,d);
true
```

24. *How can I find the quotient (factor) group of the Rubik's cube group by its commutator (derived) subgroup?*

```
gap> d:=DerivedSubgroup(rubik);
<permutation group of size 21626001637244928000 with 5 generators>

gap> f:=FactorGroup(rubik,d);
Group([ f1 ])

gap> Size(f);
2
```

25. *How can I find some Sylow p-subgroups of the Rubik's cube group?*

```
gap> Read("C:/rubik.txt");

gap> rubik:=Group(r,l,u,d,f,b);
<permutation group with 6 generators>

gap> Size(rubik);
43252003274489856000

gap> FactorsInt(43252003274489856000);
[ 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2,
2, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 5, 5, 5, 7, 7, 11 ]

gap> sylow2:=SylowSubgroup(rubik,2);
<permutation group of size 134217728 with 27 generators>

gap> sylow3:=SylowSubgroup(rubik,3);
<permutation group of size 4782969 with 14 generators>

gap> sylow5:=SylowSubgroup(rubik,5);
<permutation group of size 125 with 3 generators>

gap> sylow7:=SylowSubgroup(rubik,7);
<permutation group of size 49 with 2 generators>

gap> sylow11:=SylowSubgroup(rubik,11);
Group([ (4,36,31,39,42,12,5,21,15,13,7)(10,29,45,47,23,37,26,28,44,20,18) ])
```

```
gap> Elements(sylow11);
[ (), (4,5,36,21,31,15,39,13,42,7,12)(10,26,29,28,45,44,47,20,23,18,37),
(4,7,13,15,21,5,12,42,39,31,36)(10,18,20,44,28,26,37,23,47,45,29),
(4,12,7,42,13,39,15,31,21,36,5)(10,37,18,23,20,47,44,45,28,29,26),
(4,13,21,12,39,36,7,15,5,42,31)(10,20,28,37,47,29,18,44,26,23,45),
(4,15,12,31,7,21,42,36,13,5,39)(10,44,37,45,18,28,23,29,20,26,47),
(4,21,39,7,5,31,13,12,36,15,42)(10,28,47,18,26,45,20,37,29,44,23),
(4,31,42,5,15,7,36,39,12,21,13)(10,45,23,26,44,18,29,47,37,28,20),
(4,36,31,39,42,12,5,21,15,13,7)(10,29,45,47,23,37,26,28,44,20,18),
(4,39,5,13,36,42,21,7,31,12,15)(10,47,26,20,29,23,28,18,45,37,44),
(4,42,15,36,12,13,31,5,7,39,21)(10,23,44,29,37,20,45,26,18,47,28) ]

gap> IsNormal(rubik,sylow2);
false

gap> IsNormal(rubik,sylow3);
false


gap> IsNormal(rubik,sylow5);
false

gap> IsNormal(rubik,sylow7);
false

gap> IsNormal(rubik,sylow11);
false
```

NOTE: All of the *Sylow p-subgroups* found above have *conjugates*, but the sheer size of the *Rubik's cube group* makes it too difficult to pursue them on a typical desktop computer.

26. *How do I determine if a group is cyclic?*

```
gap> a:=(1,2,3)*(4,5,6,7);
(1,2,3)(4,5,6,7)

gap> g:=Group(a);
Group([ (1,2,3)(4,5,6,7) ])

gap> Size(g);
12

gap> IsCyclic(g);
true
```

27. *How do I create a dihedral group with 2n elements for an n-sided regular polygon?*

```
gap> d4:=DihedralGroup(IsPermGroup,8);
Group([ (1,2,3,4), (2,4) ])
```

```
gap> Elements(d4);
[ (), (2,4), (1,2)(3,4), (1,2,3,4), (1,3), (1,3)(2,4), (1,4,3,2), (1,4)(2,3) ]
```

28. *How can I express the elements of a dihedral group as rotations and flips rather than as permutations?*

```
gap> d3:=DihedralGroup(6);
<pc group of size 6 with 2 generators>

gap> Elements(d3);
[ <identity> of ..., f1, f2, f1*f2, f2^2, f1*f2^2 ]

gap> ShowMultiplicationTable(d3);
*                 | <identity> of ... f1                f2                f1*f2             f2^2              f1*f2^2
------------------+-------------------------------------------------------------------------------------------------\
---------
<identity> of ... | <identity> of ... f1                f2                f1*f2             f2^2              f1*f2^2
f1                | f1                <identity> of ... f1*f2             f2                f1*f2^2           f2^2
f2                | f2                f1*f2^2           f2^2              f1                <identity> of ... f1*f2
f1*f2             | f1*f2             f2^2              f1*f2^2           <identity> of ... f1                f2
f2^2              | f2^2              f1*f2             <identity> of ... f1*f2^2           f2                f1
f1*f2^2           | f1*f2^2           f2                f1                f2^2              f1*f2
<identity> of ...
```

29. *How do I create a symmetric group of degree n with n! elements?*

```
gap> s4:=SymmetricGroup(4);
Sym( [ 1 .. 4 ] )

gap> Size(s4);
24

gap> Elements(s4);
[ (), (3,4), (2,3), (2,3,4), (2,4,3), (2,4), (1,2), (1,2)(3,4), (1,2,3),
(1,2,3,4), (1,2,4,3), (1,2,4), (1,3,2),
  (1,3,4,2), (1,3), (1,3,4), (1,3)(2,4), (1,3,2,4), (1,4,3,2), (1,4,2), (1,4,3),
(1,4), (1,4,2,3), (1,4)(2,3) ]
```

30. *How do I create an alternating group of degree n with* $\frac{n!}{2}$ *elements?*

```
gap> a4:=AlternatingGroup(4);
Alt( [ 1 .. 4 ] )

gap> Size(a4);
12
gap> Elements(a4);
[ (), (2,3,4), (2,4,3), (1,2)(3,4), (1,2,3), (1,2,4), (1,3,2), (1,3,4),
(1,3)(2,4), (1,4,2), (1,4,3), (1,4)(2,3) ]
```

31. *How do I create a direct product of two or more groups?*

```
gap> g1:=Group((1,2,3));
Group([ (1,2,3) ])
```

```
gap> g2:=Group((4,5));
Group([ (4,5) ])

gap> dp:=DirectProduct(g1,g2);
Group([ (1,2,3), (4,5) ])

gap> Size(dp);
6
gap> Elements(dp);
[ (), (4,5), (1,2,3), (1,2,3)(4,5), (1,3,2), (1,3,2)(4,5) ]

gap> ShowMultiplicationTable(dp);
*            | ()           (4,5)        (1,2,3)      (1,2,3)(4,5) (1,3,2)
(1,3,2)(4,5)
-------------+-----------------------------------------------------------------------
---
()           | ()           (4,5)        (1,2,3)      (1,2,3)(4,5) (1,3,2)
(1,3,2)(4,5)
(4,5)        | (4,5)        ()           (1,2,3)(4,5) (1,2,3)      (1,3,2)(4,5) (1,3,2)
(1,2,3)      | (1,2,3)      (1,2,3)(4,5) (1,3,2)      (1,3,2)(4,5) ()           (4,5)
(1,2,3)(4,5) | (1,2,3)(4,5) (1,2,3)      (1,3,2)(4,5) (1,3,2)      (4,5)        ()
(1,3,2)      | (1,3,2)      (1,3,2)(4,5) ()           (4,5)        (1,2,3)
(1,2,3)(4,5)
(1,3,2)(4,5) | (1,3,2)(4,5) (1,3,2)      (4,5)        ()           (1,2,3)(4,5) (1,2,3)
```

32. *How can I create the Quaternion group?*

```
gap> a:=(1,2,5,6)*(3,8,7,4);
(1,2,5,6)(3,8,7,4)

gap> b:=(1,4,5,8)*(2,7,6,3);
(1,4,5,8)(2,7,6,3)

gap> q:=Group(a,b);
Group([ (1,2,5,6)(3,8,7,4), (1,4,5,8)(2,7,6,3) ])

gap> Size(q);
8

gap> IsAbelian(q);
false

gap> Elements(q);
[ (), (1,2,5,6)(3,8,7,4), (1,3,5,7)(2,4,6,8), (1,4,5,8)(2,7,6,3),
(1,5)(2,6)(3,7)(4,8), (1,6,5,2)(3,4,7,8),
  (1,7,5,3)(2,8,6,4), (1,8,5,4)(2,3,6,7) ]

gap> q:=QuaternionGroup(IsPermGroup,8);
Group([ (1,5,3,7)(2,8,4,6), (1,2,3,4)(5,6,7,8) ])

gap> Size(q);
8


gap> IsAbelian(q);
false

gap> Elements(q);
[ (), (1,2,3,4)(5,6,7,8), (1,3)(2,4)(5,7)(6,8), (1,4,3,2)(5,8,7,6),
(1,5,3,7)(2,8,4,6), (1,6,3,8)(2,5,4,7),
  (1,7,3,5)(2,6,4,8), (1,8,3,6)(2,7,4,5) ]
```

33. *How can I find a set of independent generators for a group?*

```
gap> c6:=CyclicGroup(IsPermGroup,6);
Group([ (1,2,3,4,5,6) ])

gap> Size(c6);
6

gap> GeneratorsOfGroup(c6);
[ (1,2,3,4,5,6) ]


gap> d4:=DihedralGroup(IsPermGroup,8);
Group([ (1,2,3,4), (2,4) ])

gap> Size(d4);
8

gap> GeneratorsOfGroup(d4);
[ (1,2,3,4), (2,4) ]


gap> s5:=SymmetricGroup(5);
Sym( [ 1 .. 5 ] )


gap> Size(s5);
120

gap> GeneratorsOfGroup(s5);
[ (1,2,3,4,5), (1,2) ]


gap> a5:=AlternatingGroup(5);
Alt( [ 1 .. 5 ] )

gap> Size(a5);
60

gap> GeneratorsOfGroup(a5);
[ (1,2,3,4,5), (3,4,5) ]


gap> q:=QuaternionGroup(IsPermGroup,8);
Group([ (1,5,3,7)(2,8,4,6), (1,2,3,4)(5,6,7,8) ])

gap> Size(q);
8

gap> GeneratorsOfGroup(q);
[ (1,5,3,7)(2,8,4,6), (1,2,3,4)(5,6,7,8) ]
```

34. *How do I find the conjugate of a permutation in the form $a^b = b^{-1}ab$?*

```
gap> a:=(1,2,3,4,5);
(1,2,3,4,5)

gap> b:=(2,4,5);
(2,4,5)
```

```
gap> a^b;
(1,4,3,5,2)

gap> b^-1*a*b;
(1,4,3,5,2)
```

35. *How do I divide up a group into classes of elements that are conjugate to one another? (Note that "conjugacy" is an equivalence relation on our group G. That means that G can be separated into nonintersecting subsets that contain only elements that are conjugate to one another.)*

```
gap> d3:=DihedralGroup(IsPermGroup,6);
Group([ (1,2,3), (2,3) ])

gap> Size(d3);
6

gap> Elements(d3);
[ (), (2,3), (1,2), (1,2,3), (1,3,2), (1,3) ]

gap> cc:=ConjugacyClasses(d3);
[ ()^G, (2,3)^G, (1,2,3)^G ]

gap> Elements(cc[1]);
[ () ]

gap> Elements(cc[2]);
[ (2,3), (1,2), (1,3) ]

gap> Elements(cc[3]);
[ (1,2,3), (1,3,2) ]
```

GROUPS OF ORDER 1

The only *group* of order 1 is the *group* that consists of a single element, the *identify element*. Consequently, it's a pretty simple *group*, and there is not much detail to give about it.

THE IDENTITY GROUP

Generators:

$( )$

Elements:

`{ () }`

Is Abelian?

Yes

Subgroups:

$\{( )\}$
Normal, Center, Commutator Subgroup

Subgroup Lattice

e

GROUPS OF ORDER 2

Just as there is only one *group* of order 1, there is also only one *group*, *up to isomorphism*, of order 2. Also, when we use the phrase *"up to isomorphism,"* that means that even though we might use different names for the elements of the *group* and even though our *binary operations* may be defined differently in the different *groups*, the resulting multiplication tables all have the same algebraic structure. That means that we can take the elements of one *group*, translate them into elements of the other *group*, and then the corresponding elements will combine with one another in the same way. For example, below are four different looking multiplication tables that all represent the one *group* of order 2 (*up to isomorphism*).

	0	**1**
0	0	1
1	1	0

	1	**2**
1	1	2
2	2	1

	a	**b**
a	a	b
b	b	a

	no flip	**flip**
no flip	no flip	flip
flip	flip	no flip

For the last *group multiplication table* in our list, what we have in mind is a light switch and the 2-element *group* associated with it. Doing nothing, not flipping the switch at all, is the *identity element* in this *group*. The only other element in the *group* is represented by flipping the switch, and if we flip the switch twice, then the result is the same as not flipping the switch at all. In other words, "flip times flip = no flip."

<u>THE CYCLIC GROUP OF ORDER 2</u>

$C_2 \cong \mathbb{Z}_2$

<u>Generators:</u>

$(1,2)$

<u>Elements:</u>

`{ (), (1,2) }`

<u>Is Abelian?</u>

Yes

<u>Subgroups:</u>

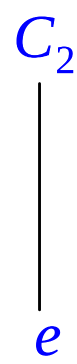
$\left\{ \begin{matrix} (\ ) \\ (1,2) \end{matrix} \right\} \cong C_2$

Normal, Center, Sylow *2*-subgroup

$\{(\ )\}$

Normal, Commutator Subgroup

Subgroup Lattice



Groups of Order 3

There is also only one *group* of order 3, and it is the *cyclic group* C_3. Notice, too, that 3 is a prime number. Whenever the order of a *group* is a prime such as 2 or 3, then the only *group* of that order is going to be a *cyclic group*. This is because for *finite groups* the order of any *subgroup* has to be a divisor of the order of the *group*, and the only divisors of a prime number are itself and 1. Hence, the only *subgroups* of a *group* of prime order are the whole *group* and the *identity*, and they are also *normal subgroups*. Furthermore, C_3 is *simple* since it doesn't have any *normal subgroups* besides itself and the *identity*. Notice, also, that for any given finite order, there always exists a *cyclic group* of that order. Hence, when the order is prime, the only *group* that exists is the *cyclic group* of that prime order.

THE CYCLIC GROUP OF ORDER 3

$C_3 \cong \mathbb{Z}_3$

Generators:

$(1,2,3)$

Elements:

```
{ (), (1,2,3), (1,3,2) }
```

Is Abelian?

Yes

Subgroups:

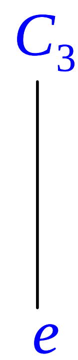
$$\left\{ \begin{matrix} (\ ) \\ (1,2,3) \\ (1,3,2) \end{matrix} \right\} \cong C_3$$

Normal, Center, Sylow *3*-subgroup

$$\{(\ )\}$$

Normal, Commutator Subgroup

Subgroup Lattice



GROUPS OF ORDER 4

There exist two *groups* of order 4 and both are *abelian*. Consequently, we can apply the *Fundamental Theorem of Finite Abelian Groups* which tells us that each *group* can be expressed as a *direct product* of *cyclic groups* of prime power order. In this case that means that the only two possible *groups* are the *cyclic group* C_4 and the *direct product* $C_2 \times C_2$. The *group* $C_2 \times C_2$ is also known as the *Klein 4-group* or as *Vierergruppe* (German for 4-*group*). Additionally, it is sometimes denoted by K_4 or by V, and a good representation for this *group* consists of two light switches each of which can be flipped on or off. Let f_1 represent flipping the first switch, let f_2 represent flipping the second switch, and let 0 represent no flip at all. Then using this notation we can represent the elements of the *group* as $\{(0,0),(f_1,0),(0,f_2),(f_1,f_2)\}$ where $f_1^2 = 0 = f_2^2$.

<u>THE CYCLIC GROUP OF ORDER 4</u>

$C_4 \cong \mathbb{Z}_4$

<u>Generators:</u>

$(1,2,3,4)$

<u>Elements:</u>

`{ (), (1,2,3,4), (1,3)(2,4), (1,4,3,2) }`

<u>Is Abelian?</u>

Yes

<u>Subgroups:</u>

$$\left\{\begin{matrix} (\ ) \\ (1,2,3,4) \\ (1,3)(2,4) \\ (4,3,2,1) \end{matrix}\right\} \cong C_4$$

Normal, Center, Sylow *2*-subgroup

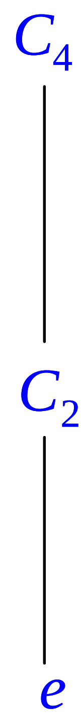
$$\left\{\begin{matrix} (\ ) \\ (1,3)(2,4) \end{matrix}\right\} \cong C_2$$

Normal

$$\{(\ )\}$$

Normal, Commutator Subgroup

Subgroup Lattice



THE KLEIN 4-GROUP

$C_2 \times C_2 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$

Generators:

$(1,2),(3,4)$

Elements:

```
{ (), (3,4), (1,2), (1,2)(3,4) }
```

Is Abelian?

Yes

Subgroups:

$$\left\{\begin{matrix} (\ ) \\ (1,2) \\ (3,4) \\ (1,2)(3,4) \end{matrix}\right\} \cong C_2 \times C_2$$

Normal, Center, Sylow *2*-subgroup

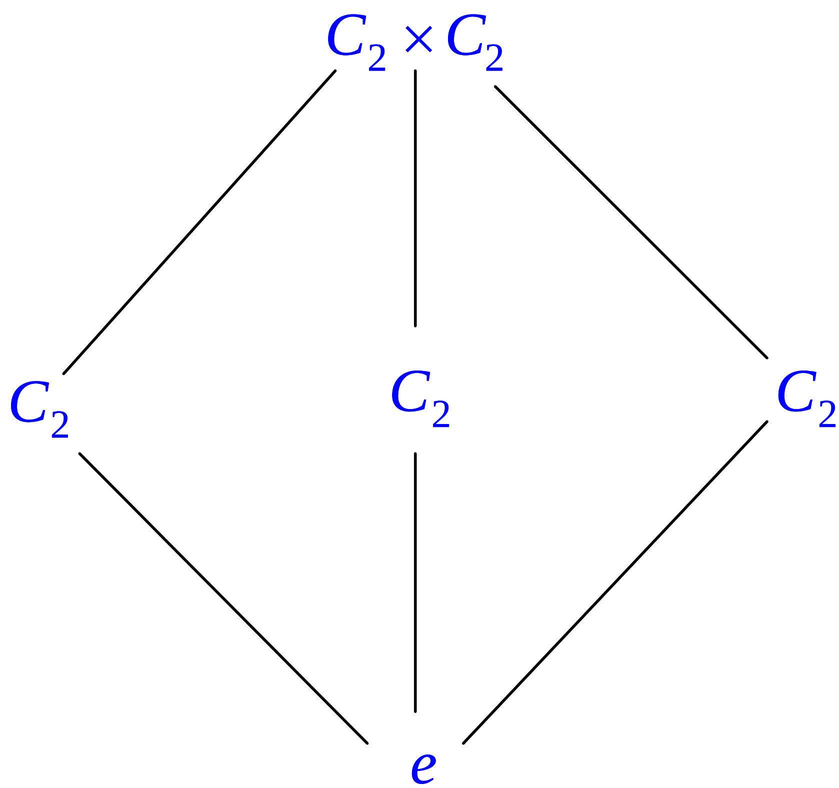
$$\left\{\begin{matrix} (\ ) \\ (1,2) \end{matrix}\right\} \qquad \left\{\begin{matrix} (\ ) \\ (3,4) \end{matrix}\right\} \qquad \left\{\begin{matrix} (\ ) \\ (1,2)(3,4) \end{matrix}\right\} \cong C_2$$

Normal Normal Normal

$\{(\ )\}$

Normal, Commutator Subgroup

Subgroup Lattice



GROUPS OF ORDER 5

Since 5 is a prime number, the only *group* that exists of order 5 is the *abelian cyclic group* of order 5, C_5. Furthermore, this *group* is simple since its only *normal subgroups* are itself and the *identity*.

THE CYCLIC GROUP OF ORDER 5

$C_5 \cong \mathbb{Z}_5$

Generators:

$(1,2,3,4,5)$

Elements:

`{ (), (1,2,3,4,5), (1,3,5,2,4), (1,4,2,5,3), (1,5,4,3,2) }`

Is Abelian?

Yes

Subgroups:

$$\left\{\begin{matrix} (\ ) \\ (1,2,3,4,5) \\ (1,3,5,2,4) \\ (1,4,2,5,3) \\ (1,5,4,3,2) \end{matrix}\right\} \cong C_5$$

Normal, Center, Sylow *5*-subgroup

$\{(\ )\}$

Normal. Commutator Subgroup

Subgroup Lattice

$$C_5$$

$$|$$

$$e$$

GROUPS OF ORDER 6

Order 6 for *groups* is worthy of note because this is the first time we encounter a *nonabelian group*! In fact, there exist just two *groups* of order 6 (two *groups* with six elements). One is the *cyclic group* of order 6, C_6, and the other is the *dihedral group* of degree 3, D_3. The *dihedral group* of degree 3 is the smallest *nonabelian group* there is, and yet it is interesting that all of its *proper subgroups* (*subgroups* not equal to the entire *group*) are *abelian*. Notice, too, that 6 is not a prime number, but that we can write 6 as 2×3 where 2 and 3 are *relatively prime* (that means that their only common factor is 1). When that happens with the order of a *cyclic group*, that means that we can also write our *cyclic group* as the *direct product* of smaller *cyclic groups* of prime power order, and in this case we can write $C_6 \cong C_3 \times C_2$. The *dihedral group* D_3 has order 6, and recall that it represents the symmetries of an equilateral triangle. In other words, it is the group generated by rotations of our triangle through angles that are integer multiples of $120°$ and by flips about any of its three axes of symmetry. Furthermore, the number of permutations that can be made of 3 objects is 6, and that means that the *symmetric group* of degree 3, S_3, which is the group of all permutations that can be made of 3 objects is essentially identical or *isomorphic* with the *dihedral group* D_3, $D_3 \cong S_3$. Additionally, this is the only time something like this happens. Since the order of D_n is $2n$ and since the order of S_n is $n! = n(n-1)(n-2)\ldots(1)$, the only time these two computations are the same is when $n = 3$. Something else worth noting is that for any value of *n* there always exists a *cyclic group* of degree *n*, and for any value $2n$ where $n \geq 3$, there is always a *dihedral group*, D_n, of that order, and for any *dihedral group* D_n it is also true that $D_n \cong C_n >\!\!\lhd C_2$. Thus, $D_3 \cong S_3 \cong C_3 >\!\!\lhd C_2$. A lot of *groups* of higher order turn out to be either *cyclic* or *dihedral*. And if we add to this list the s*ymmetric groups*, *direct products*, and *semidirect products*, then those are probably the majority of the *groups* we are likely to encounter. Things will change though when we get to order 8 and discover an interesting *group* called the *Quaternion Group* which is *nonabelian* and which falls into none of the aforementioned categories.

THE CYCLIC GROUP OF ORDER 6

$$C_6 \cong C_2 \times C_3 \cong \mathbb{Z}_2 \times \mathbb{Z}_3 \cong \mathbb{Z}_6$$

Generators:

$(1,2),(3,4,5)$

Elements:

`{ (), (3,4,5), (3,5,4), (1,2), (1,2)(3,4,5), (1,2)(3,5,4) }`

Is Abelian?

Yes

Subgroups:

$$\left\{\begin{matrix} (\ ) \\ (3,4,5) \\ (3,5,4) \\ (1,2) \\ (1,2)(3,4,5) \\ (1,2)(3,5,4) \end{matrix}\right\} \cong C_6$$

Normal, Center

$$\left\{\begin{matrix} (\ ) \\ (3,4,5) \\ (3,5,4) \end{matrix}\right\} \cong C_3$$

Normal, Sylow *3*-subgroup

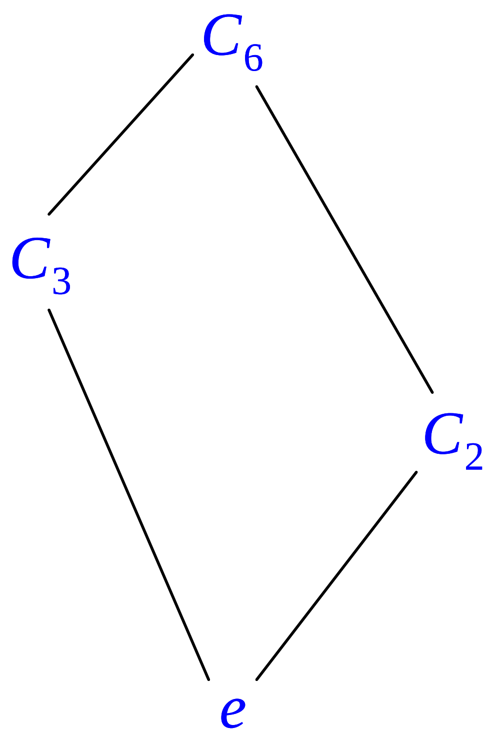
$$\left\{\begin{matrix} (\ ) \\ (1,2) \end{matrix}\right\} \cong C_2$$

Normal, Sylow *2*-subgroup

$$\{(\ )\}$$

Normal, Commutator Subgroup

Subgroup Lattice



THE DIHEDRAL/SYMMETRIC GROUP OF ORDER 6

$$D_3 \cong S_3 \cong \mathbb{Z}_3 >\!\!\lhd \mathbb{Z}_2 \cong C_3 >\!\!\lhd C_2$$

Generators:

$(1,2,3),(2,3)$

Elements:

`{ (), (2,3), (1,2), (1,2,3), (1,3,2), (1,3) }`

Is Abelian?

No

Subgroups:

$$\left\{ \begin{matrix} (\ ) \\ (1,2) \\ (1,3) \\ (2,3) \\ (1,2,3) \\ (1,3,2) \end{matrix} \right\} \cong D_3$$

Normal

$$\left\{ \begin{matrix} (\ ) \\ (1,2,3) \\ (1,3,2) \end{matrix} \right\} \cong C_3$$

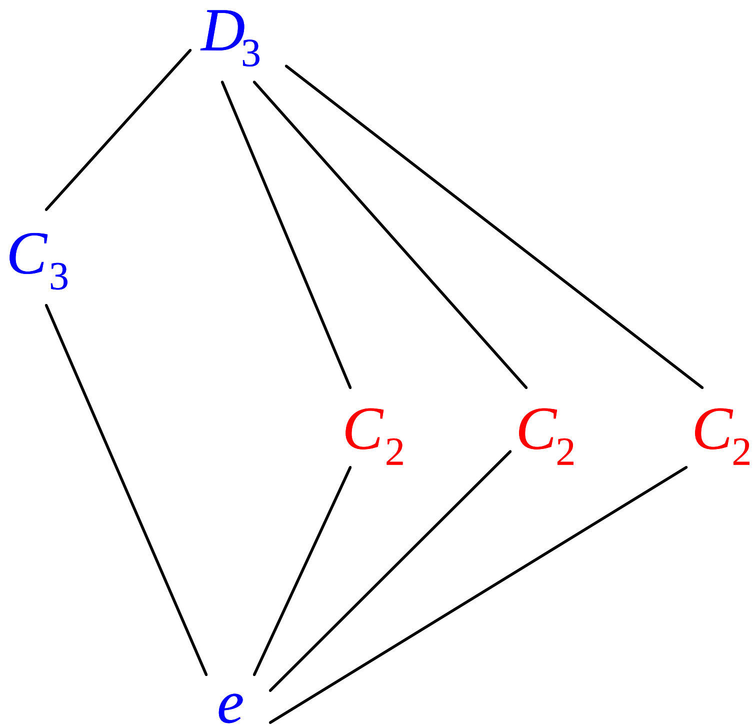
Normal, Commutator Subgroup, Sylow *3*-subgroup

$$\left\{ \begin{matrix} (\ ) \\ (1,2) \end{matrix} \right\} \qquad \left\{ \begin{matrix} (\ ) \\ (1,3) \end{matrix} \right\} \qquad \left\{ \begin{matrix} (\ ) \\ (2,3) \end{matrix} \right\} \cong C_2$$

Conjugate, Sylow *2*-subgroups

$\{(\ )\}$
Normal, Center

Subgroup Lattice



GROUPS OF ORDER 7

The number 7 is prime, so you know what that means. There exists only one *group* of order 7, and that is C_7, the *cyclic group* of order 7. Furthermore, again since 7 is prime, its only *subgroups* are itself and the *identity*.

THE CYCLIC GROUP OF ORDER 7

$C_7 \cong \mathbb{Z}_7$

Generators:

$(1,2,3,4,5,6,7)$

Elements:

{ (), (1,2,3,4,5,6,7), (1,3,5,7,2,4,6), (1,4,7,3,6,2,5), (1,5,2,6,3,7,4), (1,6,4,2,7,5,3), (1,7,6,5,4,3,2) }

Is Abelian?

Yes

Subgroups:

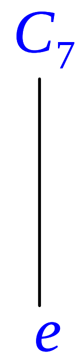
$$\left\{\begin{matrix} (\ ) \\ (1,2,3,4,5,6,7) \\ (1,3,5,7,2,4,6) \\ (1,4,7,3,6,2,5) \\ (1,5,2,6,3,7,4) \\ (1,6,4,2,7,5,3) \\ (1,7,6,5,4,3,2) \end{matrix}\right\} \cong C_7$$

Normal, Center, Sylow *7*-subgroup

$\{(\ )\}$

Normal, Commutator Subgroup

Subgroup Lattice



Groups of Order 8

Things get quite interesting once we get to 8. There exist five *groups* of order 8, and three of them are *abelian*. And by the *Fundamental Theorem of Finite Abelian Groups*, we can immediately identify the *abelian groups* as C_8, $C_4 \times C_2$, and $C_2 \times C_2 \times C_2$. Of the two *nonabelian groups*, since 8 is even we automatically know that one of them is D_4. The other *nonabelian group*, though, is called the *Quaternion Group*, and it is quite interesting since it is not one of our usual *cyclic*, *dihedral*, *symmetric*, *direct product*, or *semidirect product groups*. It is something quite different, and a notable feature of this *group* is that all of its *subgroups* are *normal* in spite of it being *nonabelian*. Also of interest is that *quaternions* were invented by the mathematician William Rowan Hamilton (1805-1865) as an extension of both *vectors* and *imaginary numbers*. Thus, we have *i, j,* and *k* which resemble the *unit vectors* studied in *trigonometry* and *advanced calculus*, and these quantities are also like *imaginary numbers* since $i^2 = j^2 = k^2 = -1$. When I was younger, *quaternions* weren't studied that much anymore, but these days there is renewed interest in the topic since they have turned out to be a useful mathematical tool for creating the kinds of computer generated effects that appear in many of today's movies.

THE CYCLIC GROUP OF ORDER 8

$C_8 \cong \mathbb{Z}_8$

Generators:

$(1,2,3,4,5,6,7,8)$

Elements:

```
{ (), (1,2,3,4,5,6,7,8), (1,3,5,7)(2,4,6,8), (1,4,7,2,5,8,3,6),
(1,5)(2,6)(3,7)(4,8), (1,6,3,8,5,2,7,4), (1,7,5,3)(2,8,6,4),
(1,8,7,6,5,4,3,2) }
```

Is Abelian?

Yes

Subgroups:

$$\left\{\begin{matrix}(\ )\\(1,2,3,4,5,6,7,8)\\(1,3,5,7)(2,4,6,8)\\(1,4,7,2,5,8,3,6)\\(1,5)(2,6)(3,7)(4,8)\\(1,6,3,8,5,2,7,4)\\(1,7,5,3)(2,8,6,4)\\(1,8,7,6,5,4,3,2)\end{matrix}\right\} \cong C_8$$

Normal, Center, Sylow *2*-subgroup

$$\left\{\begin{matrix}(\ )\\(1,3,5,7)(2,4,6,8)\\(1,5)(2,6)(3,7)(4,8)\\(1,7,5,3)(2,8,6,4)\end{matrix}\right\} \cong C_4$$

Normal

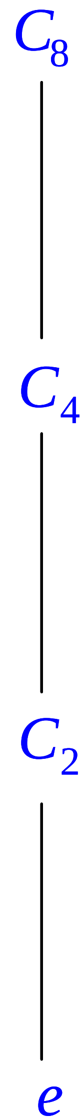
$$\left\{ \begin{array}{c} () \\ (1,5)(2,6)(3,7)(4,8) \end{array} \right\} \cong C_2$$

Normal

$$\{()\}$$

Normal, Commutator Subgroup

Subgroup Lattice



THE DIRECT PRODUCT $C_2 \times C_4$

$$C_2 \times C_4 \cong \mathbb{Z}_2 \times \mathbb{Z}_4$$

Generators:

$(1,2),(3,4,5,6)$

Elements:

{ (), (3,4,5,6), (3,5)(4,6), (3,6,5,4), (1,2), (1,2)(3,4,5,6), (1,2)(3,5)(4,6), (1,2)(3,6,5,4) }

Is Abelian?

Yes

Subgroups:

$$\begin{Bmatrix} (\ ) \\ (3,4,5,6) \\ (3,5)(4,6) \\ (3,6,5,4) \\ (1,2) \\ (1,2)(3,4,5,6) \\ (1,2)(3,5)(4,6) \\ (1,2)(3,6,5,4) \end{Bmatrix} \cong C_2 \times C_4$$

Normal, Center, Sylow *2*-subgroup

$$\begin{Bmatrix} (\ ) \\ (3,5)(4,6) \\ (1,2)(3,4,5,6) \\ (1,2)(3,6,5,4) \end{Bmatrix} \cong C_4$$

Normal

$$\left\{\begin{matrix} (\ ) \\ (3,4,5,6) \\ (3,5)(4,6) \\ (3,6,5,4) \end{matrix}\right\} \cong C_4$$

Normal

$$\left\{\begin{matrix} (\ ) \\ (3,5)(4,6) \\ (1,2) \\ (1,2)(3,5)(4,6) \end{matrix}\right\} \cong C_2 \times C_2$$

Normal

$$\left\{\begin{matrix} (\ ) \\ (1,2)(3,5)(4,6) \end{matrix}\right\} \cong C_2$$

Normal

$$\left\{\begin{matrix} (\ ) \\ (3,5)(4,6) \end{matrix}\right\} \cong C_2$$

Normal

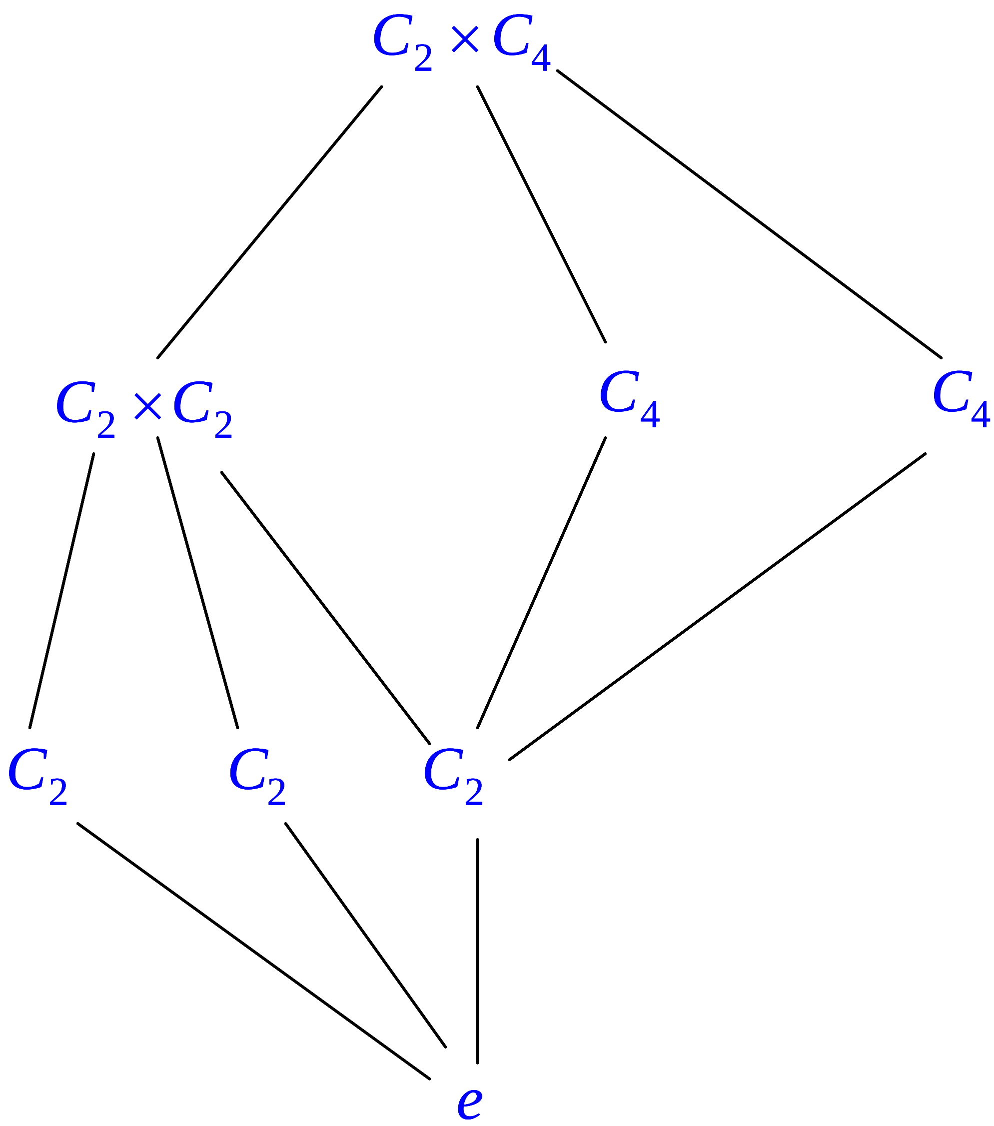
$$\left\{\begin{matrix} (\ ) \\ (1,2) \end{matrix}\right\} \cong C_2$$

Normal

$$\{(\ )\}$$

Normal, Commutator Subgroup

Subgroup Lattice



THE DIRECT PRODUCT $C_2 \times C_2 \times C_2$

$$C_2 \times C_2 \times C_2 \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$$

Generators:

$(1,2),(3,4),(5,6)$

Elements:

{ (), (5,6), (3,4), (3,4)(5,6), (1,2), (1,2)(5,6), (1,2)(3,4), (1,2)(3,4)(5,6) }

Is Abelian?

Yes

Subgroups:

$$\left\{\begin{matrix} (\ ) \\ (5,6) \\ (3,4) \\ (3,4)(5,6) \\ (1,2) \\ (1,2)(5,6) \\ (1,2)(3,4) \\ (1,2)(3,4)(5,6) \end{matrix}\right\} \cong C_2 \times C_2 \times C_2$$

Normal, Center, Sylow *2*-subgroup

$$\left\{\begin{matrix} (\ ) \\ (3,4)(5,6) \\ (1,2)(5,6) \\ (1,2)(3,4) \end{matrix}\right\} \cong C_2 \times C_2$$

Normal

$$\left\{\begin{matrix} (\ ) \\ (5,6) \\ (1,2)(3,4) \\ (1,2)(3,4)(5,6) \end{matrix}\right\} \cong C_2 \times C_2$$

Normal

$$\left\{\begin{matrix} (\ ) \\ (3,4) \\ (1,2)(5,6) \\ (1,2)(3,4)(5,6) \end{matrix}\right\} \cong C_2 \times C_2$$

Normal

$$\left\{\begin{matrix} (\ ) \\ (3,4)(5,6) \\ (1,2) \\ (1,2)(3,4)(5,6) \end{matrix}\right\} \cong C_2 \times C_2$$

Normal

$$\left\{\begin{matrix} (\ ) \\ (5,6) \\ (1,2) \\ (1,2)(5,6) \end{matrix}\right\} \cong C_2 \times C_2$$

Normal

$$\left\{\begin{matrix} (\ ) \\ (3,4) \\ (1,2) \\ (1,2)(3,4) \end{matrix}\right\} \cong C_2 \times C_2$$

Normal

$$\left\{\begin{matrix} (\ ) \\ (5,6) \\ (3,4) \\ (3,4)(5,6) \end{matrix}\right\} \cong C_2 \times C_2$$

Normal

$$\left\{\begin{matrix} (\ ) \\ (1,2)(3,4)(5,6) \end{matrix}\right\} \cong C_2$$
Normal

$$\left\{\begin{matrix} (\ ) \\ (1,2)(3,4) \end{matrix}\right\} \cong C_2$$
Normal

$$\left\{\begin{matrix} (\ ) \\ (1,2)(5,6) \end{matrix}\right\} \cong C_2$$
Normal

$$\left\{\begin{matrix} (\ ) \\ (1,2) \end{matrix}\right\} \cong C_2$$
Normal

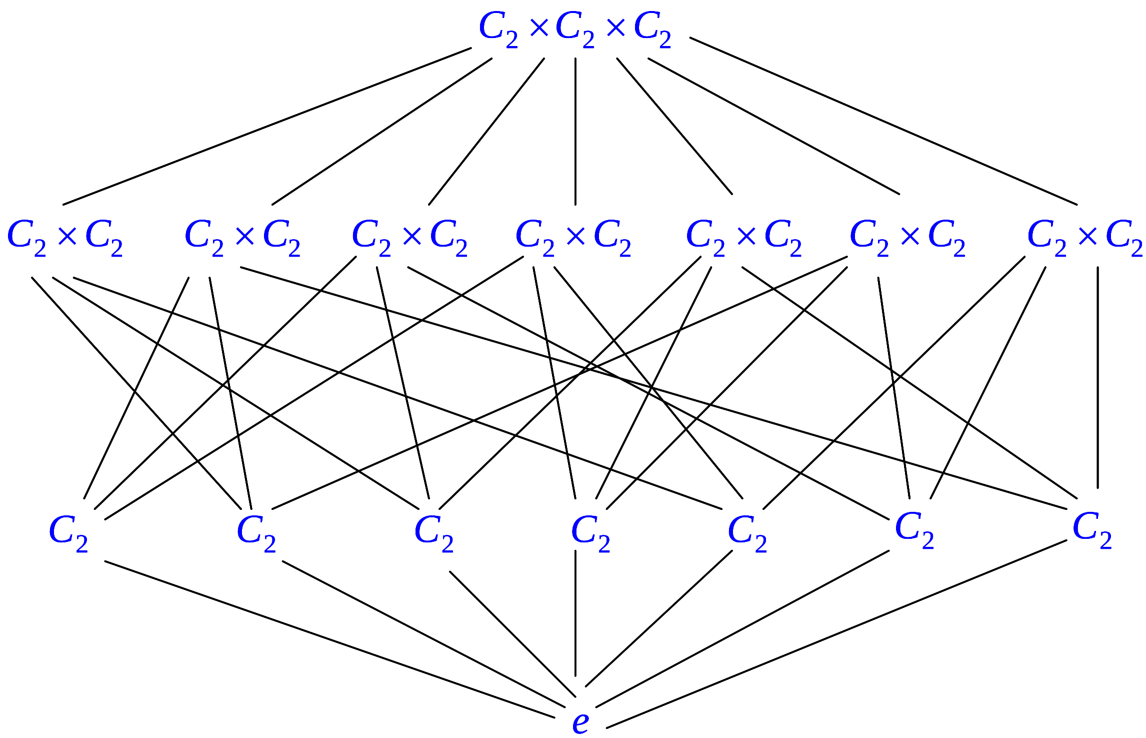
$$\left\{\begin{matrix} (\ ) \\ (3,4)(5,6) \end{matrix}\right\} \cong C_2$$
Normal

$$\left\{\begin{matrix} (\ ) \\ (3,4) \end{matrix}\right\} \cong C_2$$
Normal

$$\left\{\begin{matrix} (\ ) \\ (5,6) \end{matrix}\right\} \cong C_2$$
Normal

$$\{(\ )\}$$
Normal, Commutator Subgroup

Subgroup Lattice



THE DIHEDRAL GROUP D_4

$D_4 \cong C_4 \rtimes C_2 \cong \mathbb{Z}_4 \rtimes \mathbb{Z}_2$

Generators:

$(1,2,3,4),(2,4)$

Elements:

{ (), (2,4), (1,2)(3,4), (1,2,3,4), (1,3), (1,3)(2,4), (1,4,3,2), (1,4)(2,3) }

Is Abelian?

No

Subgroups:

$$\left\{ \begin{matrix} (\ ) \\ (2,4) \\ (1,2)(3,4) \\ (1,2,3,4) \\ (1,3) \\ (1,3)(2,4) \\ (1,4,3,2) \\ (1,4)(2,3) \end{matrix} \right\} \cong D_4$$

Normal, Sylow *2*-subgroup

$$\left\{ \begin{matrix} (\ ) \\ (1,2)(3,4) \\ (1,3)(2,4) \\ (1,4)(2,3) \end{matrix} \right\} \cong C_2 \times C_2$$

Normal

$$\left\{\begin{array}{c}(\ )\\(1,2,3,4)\\(1,3)(2,4)\\(1,4,3,2)\end{array}\right\}\cong C_4$$

Normal

$$\left\{\begin{array}{c}(\ )\\(2,4)\\(1,3)\\(1,3)(2,4)\end{array}\right\}\cong C_2\times C_2$$

Normal, Center, Commutator Subgroup

$$\left\{\begin{array}{c}(\ )\\(1,4)(2,3)\end{array}\right\} \quad \left\{\begin{array}{c}(\ )\\(1,2)(3,4)\end{array}\right\}\cong C_2$$

Conjugate

$$\left\{\begin{array}{c}(\ )\\(1,3)\end{array}\right\} \quad \left\{\begin{array}{c}(\ )\\(2,4)\end{array}\right\}\cong C_2$$

Conjugate

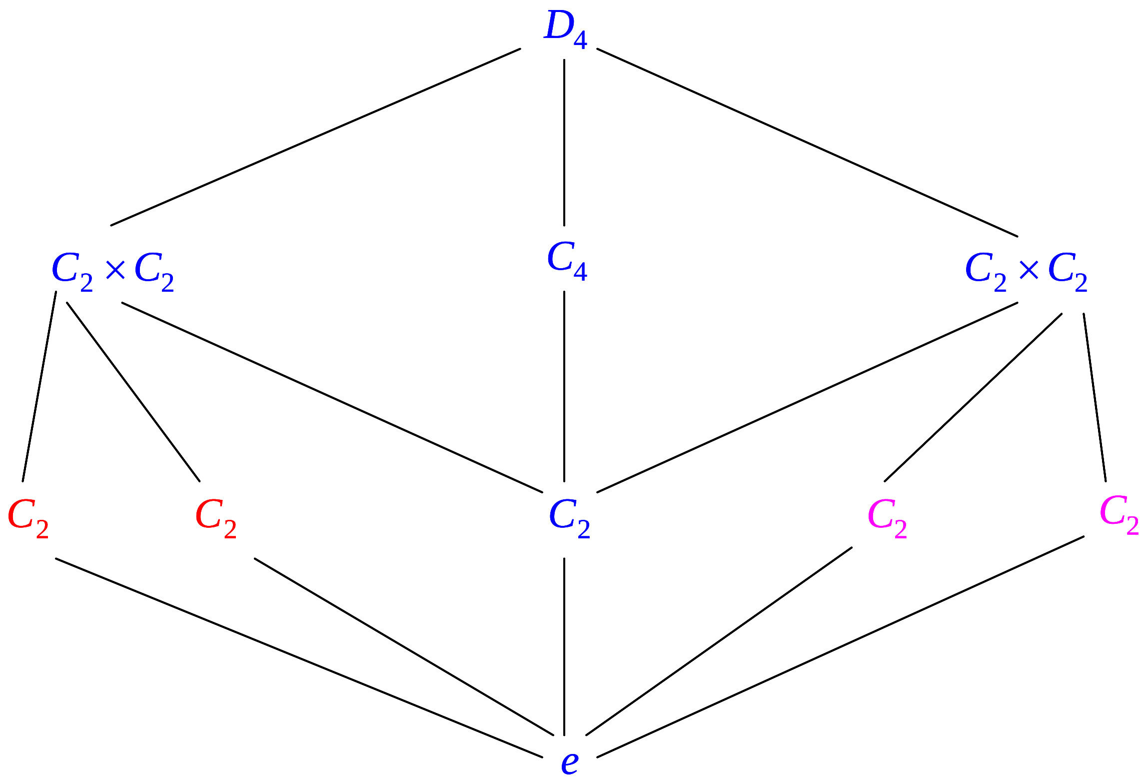
$$\left\{\begin{array}{c}(\ )\\(1,3)(2,4)\end{array}\right\}\cong C_2$$

Normal

$$\{(\ )\}$$

Normal

Subgroup Lattice



THE QUATERNION GROUP Q_8

Q_8

Generators:

$(1,2,5,6)(3,8,7,4)$, $(1,4,5,8)(2,7,6,3)$

Elements:

```
{ (), (1,2,5,6)(3,8,7,4), (1,3,5,7)(2,4,6,8), (1,4,5,8)(2,7,6,3),
(1,5)(2,6)(3,7)(4,8), (1,6,5,2)(3,4,7,8), (1,7,5,3)(2,8,6,4),
(1,8,5,4)(2,3,6,7) }
```

Is Abelian?

No

Subgroups:

$$\left\{\begin{matrix} (\ ) \\ (1,2,5,6)(3,8,7,4) \\ (1,3,5,7)(2,4,6,8) \\ (1,4,5,8)(2,7,6,3) \\ (1,5)(2,6)(3,7)(4,8) \\ (1,6,5,2)(3,4,7,8) \\ (1,7,5,3)(2,8,6,4) \\ (1,8,5,4)(2,3,6,7) \end{matrix}\right\} \cong Q_8$$

Normal, Sylow *2*-subgroup

$$\left\{\begin{matrix} (\ ) \\ (1,3,5,7)(2,4,6,8) \\ (1,5)(2,6)(3,7)(4,8) \\ (1,7,5,3)(2,8,6,4) \end{matrix}\right\} \cong C_4$$

Normal

$$\left\{\begin{matrix} (\ ) \\ (1,2,5,6)(3,8,7,4) \\ (1,5)(2,6)(3,7)(4,8) \\ (1,6,5,2)(3,4,7,8) \end{matrix}\right\} \cong C_4$$

Normal

$$\left\{\begin{matrix} (\ ) \\ (1,4,5,8)(2,7,6,3) \\ (1,5)(2,6)(3,7)(4,8) \\ (1,8,5,4)(2,3,6,7) \end{matrix}\right\} \cong C_4$$

Normal

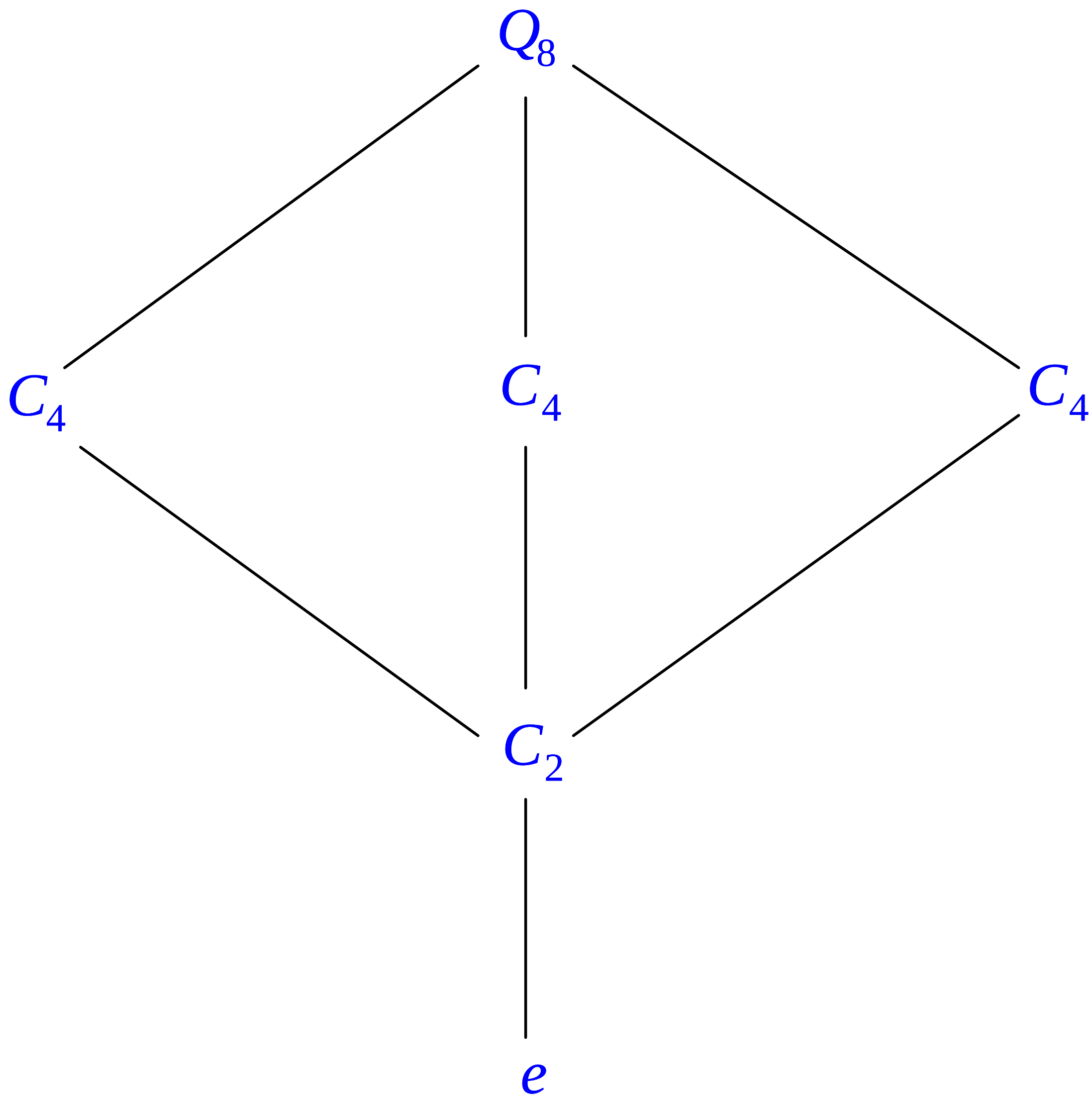
$$\left\{\begin{matrix} (\ ) \\ (1,5)(2,6)(3,7)(4,8) \end{matrix}\right\} \cong C_2$$

Normal, Center, Commutator Subgroup

$$\{(\ )\}$$

Normal

Subgroup Lattice



GROUPS OF ORDER 9

There are just two *groups* of order 9, and they are both *abelian*. One is the *cyclic group* of order 9, and the other, of course, is the *direct product* of two *cyclic groups* of order 3.

THE CYCLIC GROUP OF ORDER 9

$C_9 \cong \mathbb{Z}_9$

Generators:

$(1,2,3,4,5,6,7,8,9)$

Elements:

```
{ (), (1,2,3,4,5,6,7,8,9), (1,3,5,7,9,2,4,6,8), (1,4,7)(2,5,8)(3,6,9),
(1,5,9,4,8,3,7,2,6), (1,6,2,7,3,8,4,9,5), (1,7,4)(2,8,5)(3,9,6),
(1,8,6,4,2,9,7,5,3), (1,9,8,7,6,5,4,3,2) }
```

Is Abelian?

Yes

Subgroups:

$$\left\{\begin{matrix} (\ ) \\ (1,2,3,4,5,6,7,8,9) \\ (1,3,5,7,9,2,4,6,8) \\ (1,4,7)(2,5,8)(3,6,9) \\ (1,5,9,4,8,3,7,2,6) \\ (1,6,2,7,3,8,4,9,5) \\ (1,7,4)(2,8,5)(3,6,9) \\ (1,8,6,4,2,9,7,5,3) \\ (1,9,8,7,6,5,4,3,2 \end{matrix}\right\} \cong C_9$$

Normal, Center, Sylow *3*-subgroup

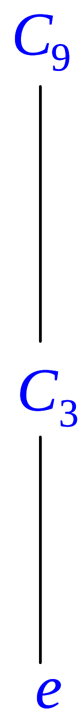
$$\left\{\begin{matrix} (\ ) \\ (1,4,7)(2,5,8)(3,6,9) \\ (1,7,4)(2,8,5)(3,9,6) \end{matrix}\right\} \cong C_3$$

Normal

$\{(\ )\}$

Normal, Commutator Subgroup

Subgroup Lattice



THE DIRECT PRODUCT $\mathbb{Z}_3 \times \mathbb{Z}_3$

$$\mathbb{Z}_3 \times \mathbb{Z}_3 \cong C_3 \times C_3$$

Generators:

$(1,2,3),(4,5,6)$

Elements:

```
{ (), (4,5,6), (4,6,5), (1,2,3), (1,2,3)(4,5,6), (1,2,3)(4,6,5), (1,3,2),
(1,3,2)(4,5,6), (1,3,2)(4,6,5) }
```

Is Abelian?

Yes

Subgroups:

$$\left\{\begin{matrix} (\ ) \\ (4,5,6) \\ (4,6,5) \\ (1,2,3) \\ (1,2,3)(4,5,6) \\ (1,2,3)(4,6,5) \\ (1,3,2) \\ (1,3,2)(4,5,6) \\ (1,3,2)(4,6,5) \end{matrix}\right\} \cong C_3 \times C_3$$

Normal, Center, Sylow *3*-subgroup

$$\left\{\begin{matrix} (\ ) \\ (1,2,3)(4,6,5) \\ (1,3,2)(4,5,6) \end{matrix}\right\} \cong C_3$$

Normal

$$\left\{\begin{matrix} (\ ) \\ (1,2,3)(4,5,6) \\ (1,3,2)(4,6,5) \end{matrix}\right\} \cong C_3$$

Normal

$$\left\{\begin{matrix} (\ ) \\ (1,2,3) \\ (1,3,2) \end{matrix}\right\} \cong C_3$$

Normal

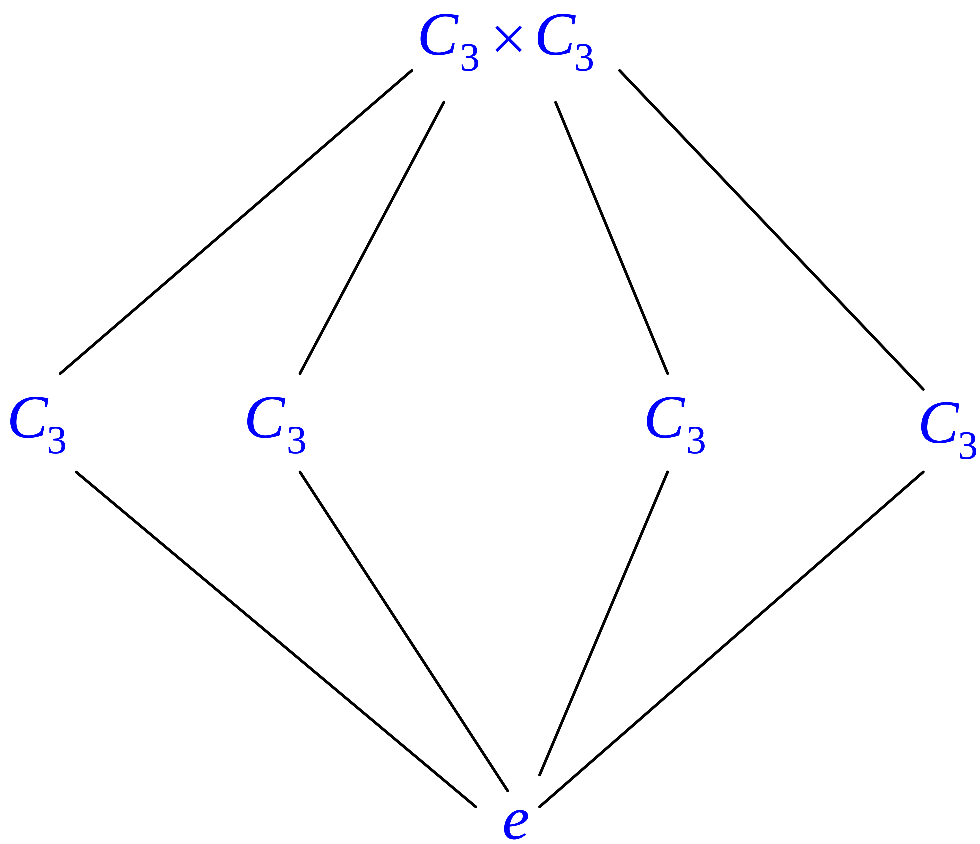
$$\left\{\begin{matrix} (\ ) \\ (4,5,6) \\ (4,6,5) \end{matrix}\right\} \cong C_3$$

Normal

$$\{(\ )\}$$

Normal, Commutator Subgroup

Subgroup Lattice



Groups of Order 10

Things are also pretty simple for *groups* of order 10. We know that one *group* of order 10 is the *abelian cyclic group* $C_{10} \cong C_5 \times C_2$, and the other is the *nonabelian* group $D_5 \cong C_5 > \triangleleft C_2$.

THE CYCLIC GROUP OF ORDER 10

$$C_{10} \cong C_2 \times C_5 \cong \mathbb{Z}_2 \times \mathbb{Z}_5 \cong \mathbb{Z}_{10}$$

Generators:

$(1,2,3,4,5,6,7,8,9,10)$

Elements:

{ (), (1,2,3,4,5,6,7,8,9,10), (1,3,5,7,9)(2,4,6,8,10), (1,4,7,10,3,6,9,2,5,8), (1,5,9,3,7)(2,6,10,4,8),(1,6)(2,7)(3,8)(4,9)(5,10), (1,7,3,9,5)(2,8,4,10,6), (1,8,5,2,9,6,3,10,7,4), (1,9,7,5,3)(2,10,8,6,4), (1,10,9,8,7,6,5,4,3,2) }

Is Abelian?

Yes

Subgroups:

$$\left\{\begin{matrix} (\ ) \\ (1,2,3,4,5,6,7,8,9,10) \\ (1,3,5,7,9)(2,4,6,8,10) \\ (1,4,7,10,3,6,9,2,5,8) \\ (1,5,9,3,7)(2,6,10,4,8) \\ (1,6)(2,7)(3,8)(4,9)(5,10) \\ (1,7,3,9,5)(2,8,4,10,6) \\ (1,8,5,2,9,6,3,10,7,4) \\ (1,9,7,5,3)(2,10,8,6,4) \\ (1,10,9,8,7,6,5,4,3,2) \end{matrix}\right\} \cong C_{10}$$

Normal, Center

$$\left\{\begin{matrix} (\ ) \\ (1,3,5,7,9)(2,4,6,8,10) \\ (1,5,9,3,7)(2,6,10,4,8) \\ (1,7,3,9,5)(2,8,4,10,6) \\ (1,9,7,5,3)(2,10,8,6,4) \end{matrix}\right\} \cong C_5$$

Normal, Sylow *5*-subgroup

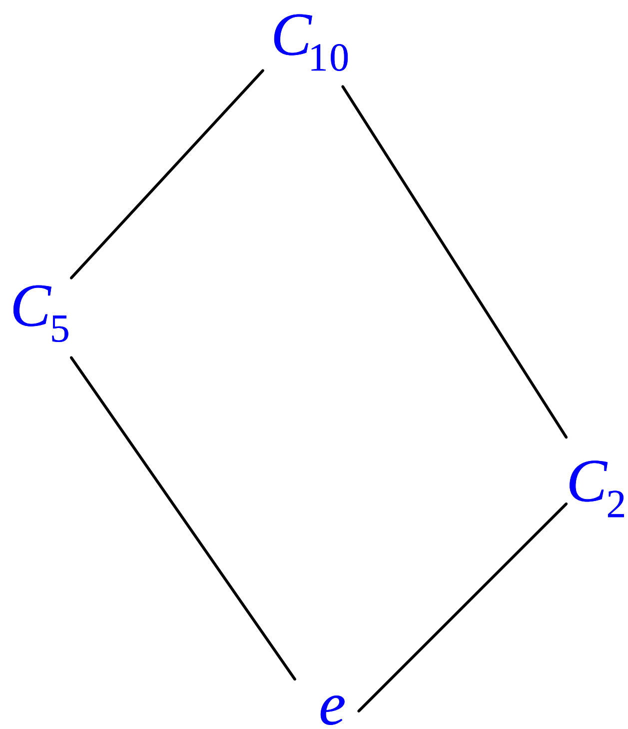
$$\left\{\begin{array}{c}(\ )\\(1,6)(2,7)(3,8)(4,9)(5,10)\end{array}\right\}\cong C_2$$

Normal, Sylow *2*-subgroup

$$\{(\ )\}$$

Normal, Commutator Subgroup

Subgroup Lattice



THE DIHEDRAL GROUP D_5

$$D_5 \cong \mathbb{Z}_5 \rtimes \mathbb{Z}_2 \cong C_5 \rtimes C_2$$

Generators:

$(1,2,3,4,5),(2,5)(3,4)$

Elements:

{ (), (2,5)(3,4), (1,2)(3,5), (1,2,3,4,5), (1,3)(4,5), (1,3,5,2,4), (1,4)(2,3), (1,4,2,5,3), (1,5,4,3,2), (1,5)(2,4) }

Is Abelian?

No

Subgroups:

$$\left\{\begin{matrix} (\ ) \\ (2,5)(3,4) \\ (1,2)(3,5) \\ (1,2,3,4,5) \\ (1,3)(4,5) \\ (1,3,5,2,4) \\ (1,4)(2,3) \\ (1,4,2,5,3) \\ (1,5,4,3,2) \\ (1,5)(2,4) \end{matrix}\right\} \cong D_5$$

Normal

$$\left\{\begin{matrix} (\ ) \\ (1,2,3,4,5) \\ (1,3,5,2,4) \\ (1,4,2,5,3) \\ (1,5,4,3,2) \end{matrix}\right\} \cong C_5$$

Normal, Commutator Subgroup, Sylow *5*-subgroup

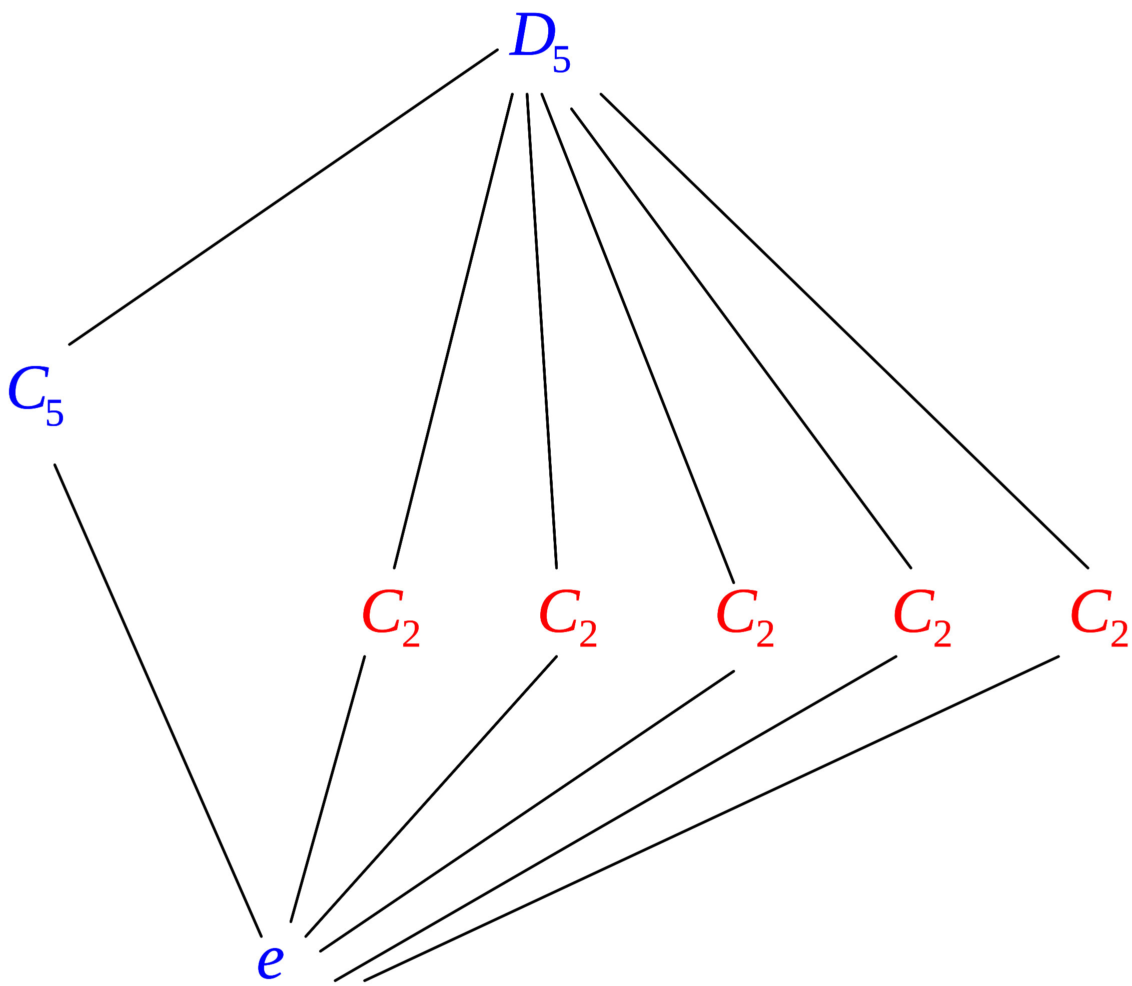
$\left\{ \begin{matrix} (\ ) \\ (1,5)(2,4) \end{matrix} \right\}$ $\left\{ \begin{matrix} (\ ) \\ (1,4)(2,3) \end{matrix} \right\}$ $\left\{ \begin{matrix} (\ ) \\ (1,3)(4,5) \end{matrix} \right\}$ $\left\{ \begin{matrix} (\ ) \\ (1,2)(3,5) \end{matrix} \right\}$ $\left\{ \begin{matrix} (\ ) \\ (2,5)(3,4) \end{matrix} \right\} \cong C_2$

Conjugate, Sylow *2*-subgroups

$\{(\ )\}$

Normal, Center

Subgroup Lattice



SUMMARY (PART 4)

In Part 4 we examined in detail the internal structure of *groups* from order 1 to order 10. You want to be very familiar with these *groups* as well as the following concepts and notations.

- *Cyclic groups,* $C_n \cong \mathbb{Z}_n$
- *Dihedral groups,* D_n
- *Symmetric groups,* S_n
- *Alternating groups,* A_n
- *Direct products,* $C_2 \times C_5$
- *Semidirect products,* $S_3 > \triangleleft\, C_2$
- *Quaternion group,* Q_8
- *Normal subgroup,* $N \triangleleft G$
- *Sylow p-subgroup*
- *Center of a group,* $Z(G)$
- *Commutator or derived subgroup*
- *Conjugate of an element,* $a^b = b^{-1}ab$
- *Conjugate of a subgroup,* $a^{-1}Ha$
- *Subgroup lattice*

More and more of the dots are being filled in!

Practice (Part 4)

Use GAP and the knowledge cleaned from the examples in this part in order to perform the following tasks.

1. *Using our analyses of the subgroup structure of groups of order 1 through 10 as a guide, complete a similar analysis for all groups of order 11.*

2. *Identify, up to isomorphism, all distinct abelian groups of order 12.*

3. *There are 2 nonabelian groups of order 12. Based upon our previous discussions in Part 3 on common types of groups, identify these two nonabelian groups.*

4. *Using our analyses of the subgroup structure of groups of order 1 through 10 as a guide, complete a similar analysis for the nonabelian groups of order 12.*

Practice (Part 4) - Answers

Use GAP and the knowledge cleaned from the examples in this part in order to perform the following tasks.

1. *Using our analyses of the subgroup structure of groups of order 1 through 10 as a guide, complete a similar analysis for all groups of order 11.*

   Since 11 is prime, the only *group* of order 11 is C_{11}.

THE CYCLIC GROUP OF ORDER 11

$C_{11} \cong \mathbb{Z}_{11}$

Generators:

$(1,2,3,4,5,6,7,8,9,10,11)$

Elements:

```
{ (), (1,2,3,4,5,6,7,8,9,10,11), (1,3,5,7,9,11,2,4,6,8,10),
(1,4,7,10,2,5,8,11,3,6,9), (1,5,9,2,6,10,3,7,11,4,8),
(1,6,11,5,10,4,9,3,8,2,7),  (1,7,2,8,3,9,4,10,5,11,6),  (1,8,4,11,7,3,10,6,2,9,5),
(1,9,6,3,11,8,5,2,10,7,4),  (1,10,8,6,4,2,11,9,7,5,3),(1,11,10,9,8,7,6,5,4,3,2) }
```

Is Abelian?

Yes

Subgroups:

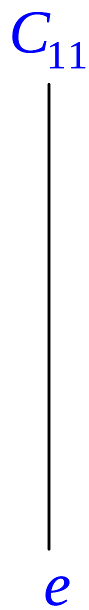
$$\left\{\begin{matrix} (\ ) \\ (1,2,3,4,5,6,7,8,9,10,11) \\ (1,3,5,7,9,11,2,4,6,8,10) \\ (1,4,7,10,2,5,8,11,3,6,9) \\ (1,5,9,2,6,10,3,7,11,4,8) \\ (1,6,11,5,10,4,9,3,8,2,7) \\ (1,7,2,8,3,9,4,10,5,11,6) \\ (1,8,4,11,7,3,10,6,2,9,5) \\ (1,9,6,3,11,8,5,2,10,7,4) \\ (1,10,8,6,4,2,11,9,7,5,3) \\ (1,11,10,9,8,7,6,5,4,3,2) \end{matrix}\right\} \cong C_{11}$$

Normal, Center, Sylow *11*-subgroup

$\{(\ )\}$

Normal, Commutator Subgroup

Subgroup Lattice



2. *Identify, up to isomorphism, all distinct abelian groups of order 12.*

Since $12 = 4 \cdot 3 = 2 \cdot 2 \cdot 3$, it follows from the *Fundamental Theorem of Finite Abelian Groups* that there are two *abelian groups* of order 12, $C_{12} \cong C_4 \times C_3$ and the direct product $C_2 \times C_2 \times C_3$.

3. *Using our analyses of the subgroup structure of groups of order 1 through 10 as a guide, complete a similar analysis for the cyclic group* C_{12}.

THE CYCLIC GROUP OF ORDER 12

$$C_{12} \cong C_3 \times C_4 \cong \mathbb{Z}_3 \times \mathbb{Z}_4 \cong \mathbb{Z}_{12}$$

Generators:

(1,2,3,4,5,6,7,8,9,10,11,12)

Elements:

```
{ (), (1,2,3,4,5,6,7,8,9,10,11,12), (1,3,5,7,9,11)(2,4,6,8,10,12),
(1,4,7,10)(2,5,8,11)(3,6,9,12), (1,5,9)(2,6,10)(3,7,11)(4,8,12),
(1,6,11,4,9,2,7,12,5,10,3,8), (1,7)(2,8)(3,9)(4,10)(5,11)(6,12),
(1,8,3,10,5,12,7,2,9,4,11,6), (1,9,5)(2,10,6)(3,11,7)(4,12,8),
(1,10,7,4)(2,11,8,5)(3,12,9,6), (1,11,9,7,5,3)(2,12,10,8,6,4),
(1,12,11,10,9,8,7,6,5,4,3,2) }
```

Is Abelian?

Yes

Subgroups:

$$\left\{\begin{matrix} (\ ) \\ (1,2,3,4,5,6,7,8,9,10,11,12) \\ (1,3,5,7,9,11)(2,4,6,8,10,12) \\ (1,4,7,10)(2,5,8,11)(3,6,9,12) \\ (1,5,9)(2,6,10)(3,7,11)(4,8,12) \\ (1,6,11,4,9,2,7,12,5,10,3,8) \\ (1,7)(2,8)(3,9)(4,10)(5,11)(6,12) \\ (1,8,3,10,5,12,7,2,9,4,11,6) \\ (1,9,5)(2,10,6)(3,11,7)(4,12,8) \\ (1,10,7,4)(2,11,8,5)(3,12,9,6) \\ (1,11,9,7,5,3)(2,12,10,8,6,4) \\ (1,12,11,10,9,8,7,6,5,4,3,2) \end{matrix}\right\} \cong C_{12}$$

Normal, Center

$$\left\{\begin{array}{c} (\ ) \\ (1,3,5,7,9,11)(2,4,6,8,10,12) \\ (1,5,9)(2,6,10)(3,7,11)(4,8,12) \\ (1,7)(2,8)(3,9)(4,10)(5,11)(6,12) \\ (1,9,5)(2,10,6)(3,11,7)(4,12,8) \\ (1,11,9,7,5,3)(2,12,10,8,6,4) \end{array}\right\} \cong C_6$$

Normal

$$\left\{\begin{array}{c} (\ ) \\ (1,4,7,10)(2,5,8,11)(3,6,9,12) \\ (1,7)(2,8)(3,9)(4,10)(5,11)(6,12) \\ (1,10,7,4)(2,11,8,5)(3,12,9,6) \end{array}\right\} \cong C_4$$

Normal, Sylow *2*-subgroup

$$\left\{\begin{array}{c} (\ ) \\ (1,5,9)(2,6,10)(3,7,11)(4,8,12) \\ (1,9,5)(2,10,6)(3,11,7)(4,12,8) \end{array}\right\} \cong C_3$$

Normal, Sylow *3*-subgroup

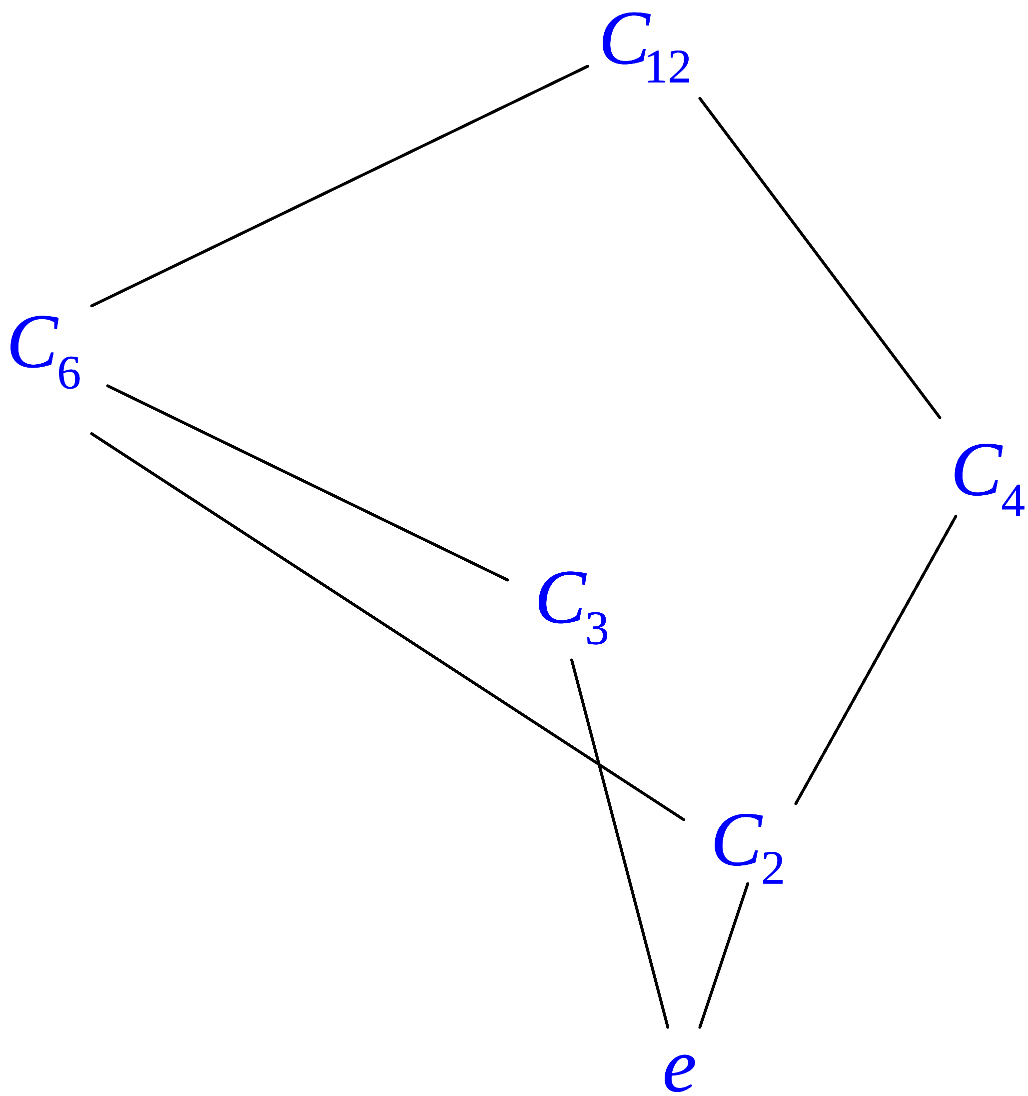
$$\left\{\begin{array}{c} (\ ) \\ (1,7)(2,8)(3,9)(4,10)(5,11)(6,12) \end{array}\right\} \cong C_2$$

Normal

$$\{(\ )\}$$

Normal, Commutator Subgroup

Subgroup Lattice



4. *There are 3 nonabelian groups of order 12. One is the semidirect product* $\mathbb{Z}_3 \rtimes \mathbb{Z}_4$*. Now, based upon our previous discussions in part 3 on common types of groups, identify the other two nonabelian groups.*

The other two *nonabelian groups* of order 12 are D_6 and A_4.

5. *Using our analyses of the subgroup structure of groups of order 1 through 10 as a guide, complete a similar analysis for the two nonabelian groups of order 12 you gave as your answer to the previous problem.*

THE DIHEDRAL GROUP D_6

$$D_6 \cong \mathbb{Z}_6 \rtimes \mathbb{Z}_2 \cong C_6 \rtimes C_2$$

Generators:

$(1,2,3,4,5,6),(2,6)(3,5)$

Elements:

{ (), (2,6)(3,5), (1,2)(3,6)(4,5), (1,2,3,4,5,6), (1,3)(4,6), (1,3,5)(2,4,6), (1,4)(2,3)(5,6), (1,4)(2,5)(3,6), (1,5)(2,4), (1,5,3)(2,6,4), (1,6,5,4,3,2), (1,6)(2,5)(3,4) }

Is Abelian?

No

Subgroups:

$$\left\{\begin{matrix} (\ ) \\ (2,6)(3,5) \\ (1,2)(3,6)(4,5) \\ (1,2,3,4,5,6) \\ (1,3)(4,6) \\ (1,3,5)(2,4,6) \\ (1,4)(2,3)(5,6) \\ (1,4)(2,5)(3,6) \\ (1,5)(2,4) \\ (1,5,3)(2,6,4) \\ (1,6,5,4,3,2) \\ (1,6)(2,5)(3,4) \end{matrix}\right\} \cong D_6$$

Normal

$$\left\{\begin{matrix} (\ ) \\ (1,2)(3,6)(4,5) \\ (1,3,5)(2,4,6) \\ (1,4)(2,3)(5,6) \\ (1,5,3)(2,6,4) \\ (1,6)(2,5)(3,4) \end{matrix}\right\} \cong D_3$$

Normal

$$\left\{\begin{matrix} (\ ) \\ (1,2,3,4,5,6) \\ (1,3,5)(2,4,6) \\ (1,4)(2,5)(3,6) \\ (1,5,3)(2,6,4) \\ (1,6,5,4,3,2) \end{matrix}\right\} \cong C_6$$

Normal

$$\left\{\begin{matrix} (\ ) \\ (2,6)(3,5) \\ (1,3,)(4,6) \\ (1,3,5)(2,4,6) \\ (1,5)(2,4) \\ (1,5,3)(2,6,4) \end{matrix}\right\} \cong D_3$$

Normal

$$\left\{\begin{matrix} (\ ) \\ (1,3)(4,6) \\ (1,4)(2,5)(3,6) \\ (1,6)(2,5)(3,4) \end{matrix}\right\} \quad \left\{\begin{matrix} (\ ) \\ (1,2)(3,6)(4,5) \\ (1,4)(2,5)(3,6) \\ (1,5)(2,4) \end{matrix}\right\} \quad \left\{\begin{matrix} (\ ) \\ (2,6)(3,5) \\ (1,4)(2,3)(5,6) \\ (1,4)(2,5)(3,6) \end{matrix}\right\} \cong C_2 \times C_2$$

Conjugate, Sylow *2*-subgroup

$$\left\{\begin{array}{c}(\ )\\(1,3,5)(2,4,6)\\(1,5,3)(1,6,4)\end{array}\right\}\cong C_3$$

Normal, Sylow *3*-subgroup

$$\left\{\begin{array}{c}(\ )\\(1,6)(2,5)(3,4)\end{array}\right\}\quad\left\{\begin{array}{c}(\ )\\(1,4)(2,3)(5,6)\end{array}\right\}\quad\left\{\begin{array}{c}(\ )\\(1,2)(3,6)(4,5)\end{array}\right\}\cong C_2$$

Conjugate

$$\left\{\begin{array}{c}(\ )\\(1,4)(2,5)(3,6)\end{array}\right\}\cong C_2$$

Normal, Commutator Subgroup, Center

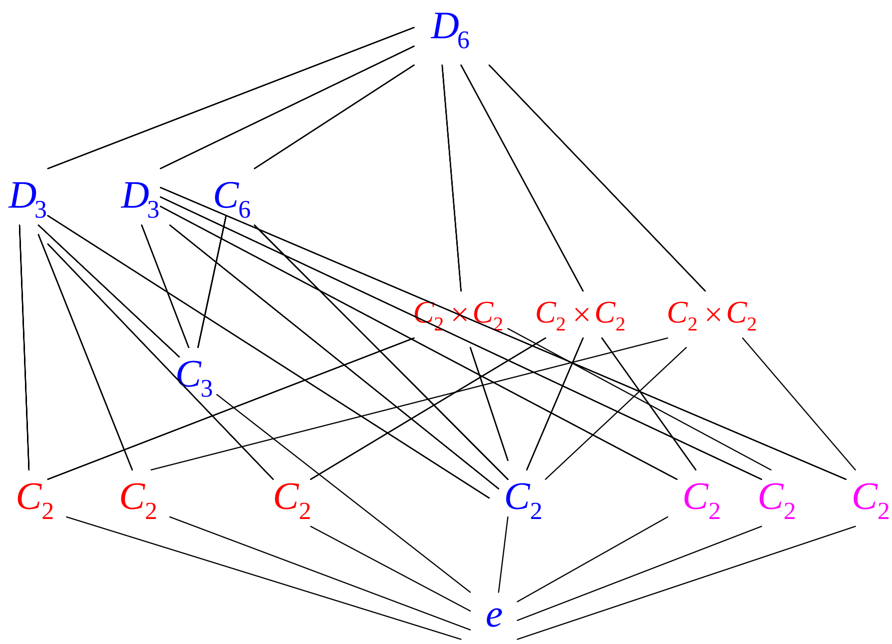
$$\left\{\begin{array}{c}(\ )\\(1,5)(2,4)\end{array}\right\}\quad\left\{\begin{array}{c}(\ )\\(1,3)(4,6)\end{array}\right\}\quad\left\{\begin{array}{c}(\ )\\(2,6)(3,5)\end{array}\right\}\cong C_2$$

Conjugate

$$\{(\ )\}$$

Normal

Subgroup Lattice



THE ALTERNATING GROUP OF DEGREE 4 A_4

A_4

Generators:

$(1,2,3),(2,3,4)$

Elements:

{ (), (2,3,4), (2,4,3), (1,2)(3,4), (1,2,3), (1,2,4), (1,3,2), (1,3,4), (1,3)(2,4), (1,4,2), (1,4,3), (1,4)(2,3) }

Is Abelian?

No

Subgroups:

$$\left\{\begin{matrix} (\ ) \\ (2,3,4) \\ (2,4,3) \\ (1,2)(3,4) \\ (1,2,3) \\ (1,2,4) \\ (1,3,2) \\ (1,3,4) \\ (1,3)(2,4) \\ (1,4,2) \\ (1,4,3) \\ (1,4)(2,3) \end{matrix}\right\} \cong A_4$$

Normal

$$\left\{\begin{matrix} (\ ) \\ (1,2)(3,4) \\ (1,3)(2,4) \\ (1,4)(2,3) \end{matrix}\right\} \cong C_2 \times C_2$$

Normal, Commutator Subgroup, Sylow *2*-subgroup

$$\begin{Bmatrix} (\ ) \\ (1,3,4) \\ (1,4,3) \end{Bmatrix} \quad \begin{Bmatrix} (\ ) \\ (1,2,4) \\ (1,4,2) \end{Bmatrix} \quad \begin{Bmatrix} (\ ) \\ (1,2,3) \\ (1,3,2) \end{Bmatrix} \quad \begin{Bmatrix} (\ ) \\ (2,3,4) \\ (2,4,3) \end{Bmatrix} \cong C_3$$

Conjugate, Sylow *3*-subgroup

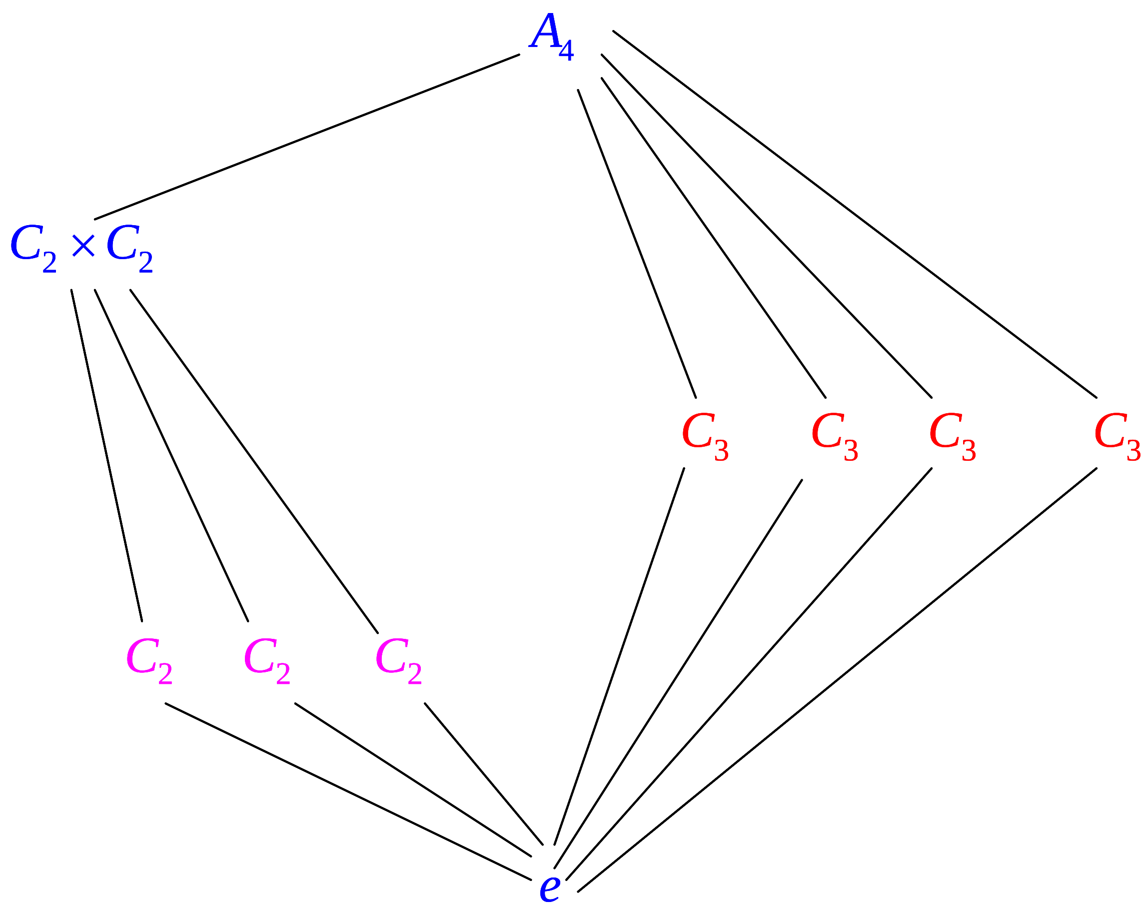
$$\begin{Bmatrix} (\ ) \\ (1,4)(2,3) \end{Bmatrix} \quad \begin{Bmatrix} (\ ) \\ (1,3)(2,4) \end{Bmatrix} \quad \begin{Bmatrix} (\ ) \\ (1,2)(3,4) \end{Bmatrix} \cong C_2$$

Conjugate

$$\{(\ )\}$$

Normal, Center

Subgroup Lattice



ALL TIME LORDS KNOW GROUP THEORY!